Constructions in Stable Commutator Length and Bounded Cohomology

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Für meine Eltern.

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Abstract

The bounded cohomology $\mathrm{H}_b^n(G, V)$ of a group G with coefficients in a normed G-module V was first systematically studied by Gromov in 1982 in his seminal paper [Gro82] in connection to the minimal volume of manifolds. Since then it has sparked much research in Geometric Group Theory. However, it is notoriously hard to explicitly compute bounded cohomology, even for the most basic groups: There is no finitely generated group G for which the full bounded cohomology $(\mathrm{H}_b^n(G, \mathbb{R}))_{n \in \mathbb{N}}$ with real coefficients is known except where it is known to vanish in all degrees (see [Mon06]). In this thesis we discuss several new constructions for classses in bounded cohomology.

There is a well-known interpretation of *ordinary* group cohomology in degrees 2 and 3 in terms of group extensions. We establish an analogous ineterpretation in the context of bounded cohomology. This involves certain maps between arbitrary groups called *quasihomomorphisms*, which were defined and studied by Fujiwara and Kapovich in [FK16].

A key open problem is to compute the full bounded cohomology $(\mathrm{H}_b^n(F, \mathbb{R}))_{n \in \mathbb{N}}$ of a non-abelian free group F with trivial real coefficients. It is known that $\mathrm{H}_b^n(F, \mathbb{R})$ is trivial for $n = 1$ and infinite dimensional for $n = 2, 3$, but essentially nothing is known about $\mathrm{H}_b^n(F, \mathbb{R})$ for $n \geq 4$. For $n = 4$, one may construct classes by taking the cup product $\alpha \smile \beta \in \mathrm{H}_b^4(F, \mathbb{R})$ between two 2-classes $\alpha, \beta \in \mathrm{H}_b^2(F, \mathbb{R})$, but it is possible that all such cup-products are trivial. We show that all such cup-products do indeed vanish if α and β are classes induced by the quasimorphisms of Brooks or Rolli.

In degree 2 there is a well-known connection between bounded cohomology and *stable commutator length (scl)* arising from Bavard's Duality Theorem. We say that a group G has a *gap in scl* if there is a $D > 0$ such that the stable commutator lenghts of any element $g \in G'$ is either zero or at least D. The maximal possible such gap is $1/2$. We develop a new criterion to tell if a group G has this maximal gap. For amalgamated free products $G = A \star_C B$ we show that every element g in the commutator subgroup of G which is not conjugate into A or B satisfies $\mathrm{scl}(g) \geq 1/2$, provided that C embeds as a *left relatively convex* subgroup in both A and B. We deduce from this that every non-trivial element g in the commutator subgroup of a right-angled Artin group G satisfies $\mathrm{scl}(g) \geq 1/2$. This bound is sharp and is inherited by the fundamental groups of all special cube complexes. We prove these statements by constructing explicit extremal homogeneous quasimorphisms $\bar{\phi} \colon G \to \mathbb{R}$ with defect at most 1 satisfying $\bar{\phi}(g) \geq 1$. Such quasimorphisms were previously unknown, even for non-abelian free groups. For these $\bar{\phi}$ there is an action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ on the circle such that $[\delta^1 \bar{\phi}] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}^2_b(G, \mathbb{R})$, where $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class; see [Ghy87].

Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known.

Chapter 3 contains material published in [Heu17b], Chapter 4 contains material published in [Heu17a] and Chapter 5 contains material published in [Heu19]. Parts of Chapter 6 contain material of [HL19], which is joint work with Clara Löh.

This thesis has not been submitted for a degree at another university.

Nicolaus Heuer, Oxford, *2nd of May 2019*

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Chapter 1

Introduction

1.1 Overview

The systematic study of bounded cohomology in geometry and group theory was initiated by Gromov in [Gro82] in connection to the minimal volume of manifolds. Since then bounded cohomology has emerged as an independent research field with many applications. These include stable commutator length ([Cal09b]), circle actions ([BFH16a]) and simplicial volume. Bounded cohomology arises naturally by "quantifying" classes in ordinary cohomology. It is a functor from groups to normed chain complexes.

A guiding theme in Geometric Group Theory is to classify groups by the geometry of the spaces they act on. In such classification schemes there are typically two extremes: amenable groups and various notions of "non-positively curved" groups; see [Bri06]. Bounded cohomology is concentrated on groups that exhibit features of negative curvature in a strong way: The bounded cohomology $\mathrm{H}^n_b(G, \mathbb{R})$ of an amenable group G with real coefficients vanishes identically in every degree $n \in \mathbb{N}$. More generally, $\mathrm{H}^n_b(\Phi) \colon \mathrm{H}^n_b(H, \mathbb{R}) \to \mathrm{H}^n_b(G, \mathbb{R})$ is an isometric isomorphism for every $n \in \mathbb{N}$ if $\Phi \colon G \to H$ is a homomorphism with amenable kernel, by *Gromov's Mapping Theorem*; see [Gro82].

Strikingly, in cases where $\mathrm{H}^n_b(G, \mathbb{R})$ is known to be non-trivial, it is typically infinite dimensional as an $\mathbb{R}$-vector space. For example $\mathrm{H}^n_b(G, \mathbb{R})$ is uncountable if $n = 2, 3$ and G is an acylindrically hyperbolic group; see [HO13], [Som97] and [FPS15]. Many "non-positively curved" groups are now known to be acylindrically hyperbolic, like non-elementary (relatively) hyperbolic groups, infinite mapping class groups of surfaces and $\mathrm{Out}(\mathbb{F}_n)$. Yet for higher degrees the bounded cohomology is utterly unknown: There is not a single finitely generated group G for which the full bounded cohomology $(\mathrm{H}_b^n(G, \mathbb{R}))_{n \in \mathbb{N}}$ with real coefficients is known, unless it is known to vanish in every degree; see [Mon06].

For a non-abelian free group F it is known that $\mathrm{H}_b^n(F, \mathbb{R})$ is trivial if $n = 1$ and of uncountable dimension in degrees $n = 2, 3$. For $n \geq 4$, $\mathrm{H}_b^n(F, \mathbb{R})$ is entirely unknown. Free groups play a distinguished rôle in constructing non-trivial classes on other acylindrically hyperbolic groups: Frigerio, Pozzetti and Sisto proved that any non-trivial alternating class in $\mathrm{H}_b^n(F, \mathbb{R})$ can be promoted to a non-trivial class in $\mathrm{H}_b^n(G, \mathbb{R})$ if G is an acylindrically hyperbolic group and $n \geq 2$; see Corollary 2 of [FPS15].

The methods available to explicitly compute bounded cohomology are usually very different from the ones used to compute ordinary group cohomology. Crucially, bounded cohomology fails to satisfy the Eilenberg–Steenrod axioms, in particular the axiom of excision. In this thesis, we study and construct several new classes in bounded cohomology.

There is a well-known interpretation of degree 2 and 3 ordinary cohomology in terms of group extensions. In Chapter 3 we shall establish an analogous interpretation in bounded cohomology. Just like ordinary cohomology, $\mathrm{H}_b^n(G, \mathbb{R})$ has the structure of a graded ring using the cup product $\smile : \mathrm{H}_b^n(G, \mathbb{R}) \times \mathrm{H}_b^m(G, \mathbb{R}) \to \mathrm{H}_b^{n+m}(G, \mathbb{R})$. For non-abelian free groups F, we will show that this cup product vanishes between many well-known classes in $\mathrm{H}_b^2(F, \mathbb{R})$ in Chapter 4. Hence to show that $\mathrm{H}_b^4(F, \mathbb{R})$ is non-trivial, new constructions are needed.

The special case of $\mathrm{H}_b^2(G, \mathbb{R})$ is often well studied and understood via the correspondence with *quasimorphisms*. Such maps may also be used to compute and estimate *stable commutator length (scl)*. We will construct several new quasimorphisms on right angled Artin groups (RAAGs) that allow us to show that non-trivial elements in the commutator subgroup of RAAGs have scl at least 1/2. This is of particular interest because this bound is inherited by all subgroups of RAAGs. The spectacular progress of recent years has revealed that this is a very rich class, following the theory of special cube complexes; see [Wis09], [HW08], [Ago13], [Bri13] and [Bri17].

Structure of this thesis

Section 1.2 of this chapter collects the main results of this thesis. In Chapter 2 we recall well-known results concerning bounded cohomology and stable commutator length.

In Chapter 3 we will study the connection of group extensions and ordinary cohomology in the setting of bounded cohomology. In Chapter 4 we will show that for non-abelian free groups the cup product between many known bounded 2-cocycles vanishes. In Chapter 5 we will construct a new type of quasimorphism on several groups including right angled Artin groups; these allow us to detect a gap of 1/2 in the stable commutator length of such groups.

Chapter 6 discusses results in collaborative work and on-going research projects.

1.2 Summary of Results

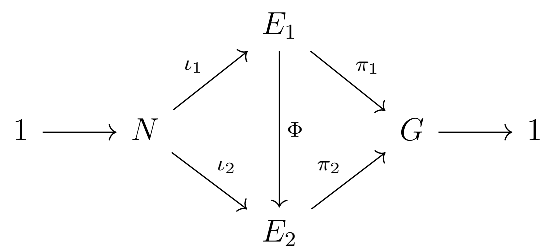
Chapter 3: Group Extensions and Bounded Cohomology

The material in this chapter is taken from [Heu17b]. For *ordinary* n-dimensional group cohomology $\mathrm{H}^n(G, V)$ with coefficients in a normed G-module V there is a well-known characterisation for $n = 2, 3$ in terms of *group extensions*. We develop an analogous characterisation for *bounded* cohomology. We first recall the classical connection between group extensions and ordinary group cohomology.

Definition. An *extension* of a group G by a group N is a short exact sequence of groups

$$1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1. \tag{1.1}$$

We say that two group extensions $1 \to N \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} G \to 1$ and $1 \to N \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} G \to 1$ of G by N are *equivalent* if there is an isomorphism $\Phi\colon E_1 \to E_2$ such that the diagram



commutes.

We will see that any group extension of G by N induces a homomorphism $\psi\colon G \to \mathrm{Out}(N)$. We denote by $\mathcal{E}(G,N,\psi)$ the set of group extensions of G by N which induce ψ under this equivalence.

There is a well-known characterisation of $\mathcal{E}(G,N,\psi)$ in terms of ordinary group cohomology.

Theorem ([Bro82, Theorem 6.6][Mac49])**.** *Let G and N be groups and let $\psi\colon G \to \mathrm{Out}(N)$ be a homomorphism. Furthermore, let $Z = Z(N)$ be the centre of N equipped with the action of G induced by ψ. Then there is a class*

$$\omega = \omega(G,N,\psi) \in \mathrm{H}^3(G,Z),$$

called obstruction*, such that $\omega = 0$ in $\mathrm{H}^3(G,Z)$ if and only if $\mathcal{E}(G,N,\psi) \neq \emptyset$. If $\mathcal{E}(G,N,\psi) \neq \emptyset$ there is a bijection between the sets $\mathrm{H}^2(G,Z)$ and $\mathcal{E}(G,N,\psi)$.*

Moreover, for a G-module Z it is possible to characterise $\mathrm{H}^3(G,Z)$ in terms of these obstructions:

Theorem ([Bro82, Section IV, 6])**.** *For any G-module Z and any $\alpha \in \mathrm{H}^3(G,Z)$ there is a group N with $Z = Z(N)$ and a homomorphism $\psi\colon G \to \mathrm{Out}(N)$ extending the action of G on Z such that $\alpha = \omega(G,N,\psi)$.*

In other words, any three dimensional class in ordinary cohomology arises as an obstruction.

We will derive analogous statements to these theorems for bounded cohomology. This will use *quasihomomorphisms* as defined and studied by Fujiwara–Kapovich in [FK16]. Let G and H be groups. A set-theoretic function $\sigma\colon G \to H$ is called *quasihomomorphism* if the set

$$D(\sigma) = \{\sigma(g)\sigma(h)\sigma(gh)^{-1} | g,h \in G\}$$

is finite.

Definition. We say that an extension $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ of G by N is *bounded*, if there is a (set theoretic) section $\sigma\colon G \to E$ such that

(i) $\sigma \colon G \to E$ is a quasihomomorphism and

(ii) the (set-theoretic) map $\phi_\sigma \colon G \to \mathrm{Aut}(N)$ induced by σ has finite image in $\mathrm{Aut}(N)$.

Here $\phi_\sigma \colon G \to \mathrm{Aut}(N)$ denotes the map $\phi_\sigma \colon g \mapsto \phi_\sigma(g)$ with

$$^{\phi_\sigma(g)}n = \iota^{-1}(\sigma(g)\iota(n)\sigma(g)^{-1}).$$

We denote by $\mathcal{E}_b(G, N, \psi)$ the set of all bounded extensions of a group G by N which induce ψ under this equivalence. This is a subset of $\mathcal{E}(G, N, \psi)$.

Analogously to the theorem for ordinary cohomology we will characterise the set $\mathcal{E}_b(G, N, \psi) \subset \mathcal{E}(G, N, \psi)$ using *bounded* cohomology.

Theorem A. *Let G and N be groups and suppose that $Z = Z(N)$, the centre of N, is equipped with a norm $\| \cdot \|$ such that $(Z, \| \cdot \|)$ has finite balls. Furthermore, let $\psi \colon G \to \mathrm{Out}(N)$ be a homomorphism with finite image.*

There is a class $\omega_b = \omega_b(G, N, \psi) \in \mathrm{H}^3_b(G, Z)$ such that $\omega_b = 0$ in $\mathrm{H}^3_b(G, Z)$ if and only if $\mathcal{E}_b(G, N, \psi) \neq \emptyset$ and $c^3(\omega_b) = 0 \in \mathrm{H}^3(G, Z)$ and if and only if $\mathcal{E}(G, N, \psi) \neq \emptyset$. If $\mathcal{E}_b(G, N, \psi) \neq \emptyset$, there is a bijection between the sets $\mathrm{H}^2(G, Z)$ and $\mathcal{E}(G, N, \psi)$ which restricts to a bijection between $im(c^2) \subset \mathrm{H}^2(G, Z)$ and $\mathcal{E}_b(G, N, \psi) \subset \mathcal{E}(G, N, \psi)$.

Here, $c^n \colon \mathrm{H}^n_b(G, Z) \to \mathrm{H}^n(G, Z)$ denotes the *comparision map*; see Subsection 2.2. We say that a normed group or module $(Z, \| \cdot \|)$ has finite balls if for every $K > 0$ the set $\{z \in Z \mid \|z\| \leq K\}$ is finite.

Just as in the case of ordinary cohomology we may ask which elements of $\mathrm{H}^3_b(G, Z)$ may be realised by obstructions. For a G-module Z we define the following subset of $\mathrm{H}^3_b(G, Z)$:

$$\mathcal{F}(G, Z) := \{\Phi^*\alpha \in \mathrm{H}^3_b(G, Z) \mid \Phi \colon G \to M, \alpha \in \mathrm{H}^3(M, Z)\}$$

where $\Phi \colon G \to M$ is a homomorphism to a *finite* group M and where $\Phi^*\alpha$ denotes the pullback of α via the homomorphism Φ. As M is finite, α is necessarily bounded. Analogously to the classical theorem we will show:

Theorem B. *Let G be a group, let Z be a normed G-module with finite balls such that G acts on Z via finitely many automorphisms. Then*

$$\{\omega_b(G, N, \psi) \in \mathrm{H}^3_b(G, Z) \mid Z = Z(N) \text{ and } \psi \text{ induces the action on } G\} = \mathcal{F}(G, Z)$$

as subsets of $\mathrm{H}^3_b(G, Z)$.

Chapter 4: Cup Product between between 2-cycles in the free group

The material in this chapter is taken from [Heu17a]. In this chapter we exclusively focus on the bounded cohomology of non-abelian free groups F with real coefficients. As mentioned in the Overview (Section 1.1), $\mathrm{H}^n_b(F, \mathbb{R})$ is entirely unknown for any $n \geq 4$ and any non-trivial classes in $\mathrm{H}^n_b(F, \mathbb{R})$ may be promoted to non-trivial classes in $\mathrm{H}^n_b(G, \mathbb{R})$ for G any non-elementary acylindrically hyperbolic group.

It is well-known that non-trivial *quasimorhisms* $\phi \colon G \to \mathbb{R}$ induce non-trivial classes $[\delta^1 \phi] \in \mathrm{H}^2_b(F, \mathbb{R})$. There are several explicit constructions for infinite families of linearly independent quasimorphisms $\phi \colon F \to \mathbb{R}$. Two prominent such families are the quasimorphisms defined by Brooks and Rolli; see Examples 2.2.6 and 2.2.7. One may hope to construct non-trivial classes in $\mathrm{H}^4_b(F, \mathbb{R})$ by taking the cup product $[\delta^1 \phi] \smile [\delta^1 \psi] \in \mathrm{H}^4_b(F, \mathbb{R})$ between two such quasimorphisms $\phi, \psi \colon F \to \mathbb{R}$. We will show that this approach fails.

Theorem C. *Let $\phi, \psi \colon F \to \mathbb{R}$ be two quasimorphisms on a non-abelian free group F where ϕ and ψ are either Brooks counting quasimorphisms on a non self-overlapping word or quasimorphisms in the sense of Rolli. Then*

$$[\delta^1 \phi] \smile [\delta^1 \psi] \in \mathrm{H}^4_b(F, \mathbb{R})$$

is trivial.

Theorem C will follow from a more general vanishing theorem. For this, we will define two new classes of quasimorphisms on free groups, namely Δ-decomposable and Δ-continuous quasimorphisms, where Δ is a certain type of operator, defined in this chapter, called *decomposition.* Each Brooks and Rolli quasimorphism will be both Δ-decomposable and Δ-continuous with respect to some decomposition Δ. We will show:

Theorem D. *Let Δ be a decomposition of F and let $\phi, \psi \colon F \to \mathbb{R}$ be quasimorphisms such that ϕ is Δ-decomposable and ψ is Δ-continuous. Then*

$$[\delta^1 \phi] \smile [\delta^1 \psi] \in \mathrm{H}^4_b(F, \mathbb{R})$$

is trivial.

It was shown by Grigorchuk [Gri95] that Brooks quasimorphisms are *dense* in the vector space of quasimorphisms in the topology of pointwise convergence. However, this topology does not allow one to deduce a general vanishing of the cup product. We therefore ask:

Question 1.2.1. Let F be a non-abelian free group. Is the cup product

$$\smile \colon \mathrm{H}^2_b(F, \mathbb{R}) \times \mathrm{H}^2_b(F, \mathbb{R}) \to \mathrm{H}^4_b(F, \mathbb{R})$$

trivial?

We mention that the cup product on bounded cohomology for other groups need not be trivial.

Chapter 5: Gaps in scl for RAAGs

The material in this chapter is taken from [Heu19]. For a group G let G' be the commutator subgroup. For an element $g \in G'$ the *commutator length* ($\mathrm{cl}(g)$) denotes the minimal number of commutators needed to express g as their product. We define the *stable commutator length (scl)* as $\mathrm{scl}(g) = \lim_{n \to \infty} \mathrm{cl}(g^n)/n$.

Bavard's Duality Theorem gives a strong link between stable commutator length and degree 2 bounded cohomology. We have already seen that certain classes in $\mathrm{H}^2_b(G, \mathbb{R})$ are represented by the coboundary of quasimorphisms $\phi \colon G \to \mathbb{R}$. On the other hand, for an element $g \in G'$,

$$\mathrm{scl}(g) = \sup_{\bar{\phi} \in \mathcal{Q}(G)} \frac{\bar{\phi}(g)}{2D(\bar{\phi})}$$

where $\mathcal{Q}(G)$ is the space of *homogeneous quasimorphisms* and $D(\bar\phi)$ is the *defect of* $\bar\phi$. Though it is known that for every element $g \in G'$ the supremum in Bavard's Duality Theorem is obtained by so-called *extremal quasimorphism*, these maps have previously been known explicitly only in special cases.

A group G is said to have a *gap in stable commutator length* if there is a constant $C > 0$ such that either $\mathrm{scl}(g) = 0$ or $\mathrm{scl}(g) \geq C$ for every non-trivial $g \in G'$. If G is non-abelian, such a constant necessarily satisfies $C \leq 1/2$. Similarly we may define gaps in scl for classes of groups. We will see that many classes of "non-positively curved" groups have a gap in scl. Having a gap in stable commutator length may be used as an obstruction to group embeddings.

In the first part of this chapter, we will construct an infinite family of extremal quasimorphisms on non-abelian free groups. Let $\mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle$ be the free group on generators $\mathtt{a}$ and $\mathtt{b}$ and let $w \in \mathbb{F}_2$ be such that it is not conjugate into $\langle \mathtt{a} \rangle$ or $\langle \mathtt{b} \rangle$. Then we will construct a homogeneous quasimorphism $\bar\phi$ such that $\bar\phi(w) \geq 1$ and $D(\bar\phi) \leq 1$. This realises the well-known gap of $1/2$ in the case of non-abelian free groups.

Our approach is as follows: instead of constructing more complicated quasimorphisms $\bar\phi$ we first "simplify" the element w. This simplification is formalised by functions $\Phi \colon G \to \mathcal{A} \subset \mathbb{F}_2$, called *letter-quasimorphisms*. Here $\mathcal{A}$ denotes the set of *alternating words* in $\mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle$ with the generators $\mathtt{a}$ and $\mathtt{b}$. These are words where each letter alternates between $\{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\{\mathtt{b}, \mathtt{b}^{-1}\}$. Letter-quasimorphisms are a special case of quasimorphisms between arbitrary groups defined by Hartnick–Schweitzer [HS16]. We show:

Theorem E. *Let G be a group, $g \in G$ and suppose that there is a letter-quasimorphism $\Phi \colon G \to \mathcal{A}$ such that $\Phi(g)$ is non-trivial and $\Phi(g^n) = \Phi(g)^n$ for all $n \in \mathbb{N}$. Then there is an explicit homogeneous quasimorphism $\bar\phi \colon G \to \mathbb{R}$ such that $\bar\phi(g) \geq 1$ and $D(\bar\phi) \leq 1$.*

If G is countable then there is an action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1 \bar\phi] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}_b^2(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.

By Bavard's Duality Theorem, it is immediate that if such an element g additionally lies in G', the commutator subgroup of G, then $\mathrm{scl}(g) \geq 1/2$. Many groups G have the property that for any element $g \in G'$ there is a letter-quasimorphism

$\Phi_g\colon G \to \mathcal{A}$ such that $\Phi_g(g^n) = \Phi_g(g)^n$ where $\Phi_g(g) \in \mathcal{A}$ is non-trivial. We will see that residually free groups and right-angled Artin groups have this property. Note the similarities of this property with being *residually free*.

In the second part of the chapter we apply Theorem E to amalgamated free products using left-orders. A subgroup $H < G$ is called *left-relatively convex* if there is an order on the left cosets G/H which is invariant under left multiplication by G. We will construct letter-quasimorphisms $G \to \mathcal{A} \subset \mathbb{F}_2$ using the sign of these orders. We deduce:

Theorem F. *Let A, B, C be groups, $\kappa_A\colon C \hookrightarrow A$ and $\kappa_B\colon C \hookrightarrow B$ injections and suppose both $\kappa_A(C) < A$ and $\kappa_B(C) < B$ are left-relatively convex. If $g \in A \star_C B$ does not conjugate into one of the factors then there is a homogeneous quasimorphism $\bar{\phi}\colon A \star_C B \to \mathbb{R}$ such that $\bar{\phi}(g) \geq 1$ and $D(\bar{\phi}) \leq 1$. If G is countable then there is an action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1 \bar{\phi}] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}^2_b(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.*

Again by Bavard's Duality Theorem we deduce that any such g which also lies in the commutator subgroup satisfies $\mathrm{scl}(g) \geq 1/2$. We apply this to right-angled Artin groups using the work of [ADS15]. This way we prove:

Theorem G. *Every non-trivial element $g \in G'$ in the commutator subgroup of a right-angled Artin group G satisfies $\mathrm{scl}(g) \geq 1/2$. This bound is sharp.*

Every subgroup of a right-angled Artin group will inherit this bound. Such groups are now known to be an extremely rich class, following the theory of special cube complexes. See [Wis09], [HW08], [Ago13], [Bri13] and [Bri17]. Stable commutator length may serve as an invariant to distinguish virtually special from special cube complexes.

Chapter 2

Preliminaries

Sections 2.2, 2.3 and 2.4 give a brief and general introduction to bounded cohomology, stable commutator length, and simplicial volume, respectively.

2.1 Notations and conventions

Throughout this thesis, Roman capitals (A, B) denote groups, lowercase Roman letters (a, b) denote group elements, greek letters (α, β) denote functions and curly capitals ($\mathcal{A}, \mathcal{B}$) denote sets. Non-abelian free groups will be denoted by F or by $\mathbb{F}_n$ to emphasise that the group is freely generated by n elements. Free generators are called letters and denoted in code-font ($\mathtt{a}, \mathtt{b}$). In a group G the identity will be denoted by $1 \in G$, or by $0 \in G$ to stress that G is abelian. The commutator subgroup of G will be denoted by G'. The trivial group will also be denoted by "1". We stick to these notations unless there is a standard mathematical convention to do otherwise.

2.2 Bounded Cohomology

Bounded cohomology of discrete groups and topological spaces was first systematically studied by Gromov [Gro82]. Gromov established the fundamental properties of bounded cohomology using so-called multicomplexes. Later, Ivanov developed a more algebraic framework via resolutions [Iva85].

The standard reference that we draw on in this introduction is the recent book by Frigerio [Fri17].

We define the bounded cohomology of groups and spaces in Subsections 2.2.1 and Subsection 2.2.2 respectively. The strong relationship between both is discussed in Subsection 2.2.3. For degree 2 we recall the correspondence to circle actions (Subsection 2.2.4) and quasimorphisms (Subsection 2.2.5).

2.2.1 Bounded cohomology of groups: Basic Definitions

We define (bounded) cohomology of group using the *inhomogeneous* resolution. Let G be a group and let V be a $\mathbb{Z}G$-module. In what follows we may refer to a $\mathbb{Z}G$-module simply as G-module. Following [Fri17], a *norm* on a G-module V is a map $\|\cdot\|\colon V \to \mathbb{R}^{\geq 0}$ such that

- $\|v\| = 0$ if and only if $v = 0$,
- $\|rv\| \leq |r|\|v\|$ for every $r \in \mathbb{Z}$, $v \in V$,
- $\|v + w\| \leq \|v\| + \|w\|$ and
- $\|gv\| = \|v\|$ for every $g \in G$, $v \in V$.

Suppose that the G-module V is equipped with a norm $\|\cdot\|$. Set $C^0(G,V) = C^0_b(G,V) = V$ and set for $n \geq 1$, $C^n(G,V) = \{\alpha\colon G^n \to V\}$. For an element $\alpha \in C^n(G,V)$ we define $\|\alpha\| = \sup_{(g_1,\ldots,g_n)\in G^n} \|\alpha(g_1,\ldots,g_n)\|$ when the supremum exists and set $\|\alpha\| = \infty$, else. For $n \geq 1$ define the *bounded chains* as

$$C^n_b(G,V) = \{\alpha \in C^n(G,V) \mid \|\alpha\| < \infty\}.$$

We define $\delta^n\colon C^n(G,V) \to C^{n+1}(G,V)$, the *coboundary operator*, as follows: Set $\delta^0\colon C^0(G,V) \to C^1(G,V)$ via $\delta^0(v)\colon g_1 \mapsto g_1 \cdot v - v$ and for $n \geq 1$ define $\delta^n\colon C^n(G,V) \to C^{n+1}(G,V)$ via

$$\begin{aligned}\delta^n(\alpha)\colon (g_1,\ldots g_{n+1}) \mapsto\; & g_1 \cdot \alpha(g_2,\ldots,g_{n+1}) \\ & + \sum_{i=1}^{n}(-1)^i \alpha(g_1,\ldots,g_i g_{i+1},\ldots,g_{n+1}) \\ & + (-1)^{n+1}\alpha(g_1,\ldots,g_n).\end{aligned}$$

Note that δ^n restricts to a map $C^n_b(G,V) \to C^{n+1}_b(G,V)$ for any $n \geq 0$. By abuse of notation we denote this restriction by δ^n as well.

It is well-known that $(C^*(G,V),\delta^*)$ is a cochain complex. The *cohomology of* G *with coefficients in* V is the homology of this complex and denoted by $\mathrm{H}^*(G,V)$. Similarly $(C_b^*(G,V),\delta^*)$ is a cochain complex and its homology is the *bounded cohomology of* G *with coefficients in* V and denoted by $\mathrm{H}_b^*(G,V)$. The inclusion map $C_b^n(G,V) \hookrightarrow C^n(G,V)$ is a chain map and induces the *comparison map* $c^n\colon \mathrm{H}_b^n(G,V) \to \mathrm{H}^n(G,V)$ on the level of cohomology. Elements in the kernel of the comparison map are called *exact classes*.

Let W be a normed H-module and let $\Phi\colon G \to H$ be a homomorphism. Denote by V the normed abelian group W equipped with G-module structure induced by Φ. We then obtain a map $\mathrm{H}^n(\Phi)\colon \mathrm{H}^n(H,W) \to \mathrm{H}^n(G,V)$ via $\mathrm{H}^n(\Phi)\colon \alpha \mapsto \Phi^*\alpha$ where $\Phi^*\alpha$ denotes the pullback of α via Φ. Similarly we obtain a map $\mathrm{H}_b^n(\Phi)\colon \mathrm{H}_b^n(H,W) \to \mathrm{H}_b^n(G,V)$.

The *cup product* is a map $\smile\colon \mathrm{H}^n(G,\mathbb{R}) \times \mathrm{H}^m(G,\mathbb{R}) \to \mathrm{H}^{n+m}(G,\mathbb{R})$ defined by setting $([\omega_1],[\omega_2]) \mapsto [\omega_1] \smile [\omega_2]$ where $[\omega_1] \smile [\omega_2] \in \mathrm{H}^{n+m}(G,\mathbb{R})$ is represented by the cocycle $\omega_1 \smile \omega_2 \in C^{n+m}(G,\mathbb{R})$ defined by setting

$$\omega_1 \smile \omega_2\colon (g_1,\dots,g_n,g_{n+1},\dots,g_{n+m}) \mapsto \omega_1(g_1,\dots,g_n)\cdot\omega_2(g_{n+1},\dots,g_{n+m}).$$

It is easy to check that this map induces a well-defined map

$$\smile\colon \mathrm{H}_b^n(G,\mathbb{R}) \times \mathrm{H}_b^m(G,\mathbb{R}) \to \mathrm{H}_b^{n+m}(G,\mathbb{R}).$$

Note that the cup product equips $\mathrm{H}^n(G,\mathbb{R})_{n\in\mathbb{N}}$ and $\mathrm{H}_b^n(G,\mathbb{R})_{n\in\mathbb{N}}$ with the structure of a graded ring.

2.2.2 Bounded cohomology of spaces

To define the (bounded) cohomology of spaces we restrict ourselves to real or integer coefficients i.e. $V = \mathbb{R}$ or $V = \mathbb{Z}$. Let X be a topological space and let $S_n(X)$ be the set of singular n-simplices in X. Moreover, let $C^n(X,V)$ be the set of maps from $S_n(X)$ to V. For an element $\alpha \in C^n(X,V)$ we set

$$\|\alpha\|_\infty := \sup\big\{|\alpha(\sigma)| \;\big|\; \sigma \in S_n(X)\big\} \in [0,\infty]$$

and let $C_b^n(X,V) \subset C^n(X,V)$ be the subset of elements that are bounded with respect to this norm. Let $\delta^n\colon C_b^n(X,V) \to C_b^{n+1}(X,V)$ be the restriction of the singular coboundary map to bounded cochains. Then the *bounded cohomology* $\mathrm{H}_b^n(X, V)$ *of* X *with coefficients in* V is the cohomology of the complex $(C_b^\bullet(X, V), \delta^\bullet)$ and denoted by $\mathrm{H}_b^n(X, V)$. For $\alpha \in \mathrm{H}_b^n(X, V)$ we define

$$\|\alpha\| = \inf\{\|\beta\|_\infty \mid \beta \in C_b^n(X, V),\ \delta^n\beta = 0, [\beta] = \alpha \in \mathrm{H}_b^n(X, V)\}$$

and observe that $\|\cdot\|$ is a semi-norm on $\mathrm{H}_b^n(X, V)$. The bounded cohomology of spaces is also functorial in both spaces and coefficients.

2.2.3 Relationship between bounded cohomology of groups and spaces

Analogously to ordinary group cohomology, also bounded cohomology of groups may be computed using classifying spaces.

Theorem 2.2.1 ([Fri17, Theorem 5.5]). *Let* X *be a model of the classifying space* BG *of the group* G*. Then* $\mathrm{H}_b^\bullet(X, \mathbb{R})$ *is canonically isometrically isomorphic to* $\mathrm{H}_b^\bullet(G, \mathbb{R})$*.*

Remarkably, this statement holds true much more generally: every topological space with the correct fundamental group can be used to compute bounded cohomology of groups; moreover, bounded cohomology ignores amenable kernels.

Theorem 2.2.2 ([Fri17, Theorem 5.8]). *Let* X *be a path-connected space. Then* $\mathrm{H}_b^\bullet(X, \mathbb{R})$ *is canonically isometrically isomorphic to* $\mathrm{H}_b^\bullet(\pi_1(X), \mathbb{R})$*.*

Theorem 2.2.3 (mapping theorem; [Fri17, Corollary 5.11]). *Let* $f\colon X \longrightarrow Y$ *be a continuous map between path-connected topological spaces. If the induced homomorphism* $\pi_1(f)\colon \pi_1(X) \longrightarrow \pi_1(Y)$ *is surjective and has amenable kernel, then* $\mathrm{H}_b^\bullet(f, \mathbb{R})\colon \mathrm{H}_b^\bullet(Y, \mathbb{R}) \longrightarrow \mathrm{H}_b^\bullet(X, \mathbb{R})$ *is an isometric isomorphism.*

In particular, the bounded cohomology of a group G may be computed as the bounded cohomology of its representation complex.

2.2.4 Bounded 2-Cocycles via Actions on the Circle and Vice Versa

This subsection states a classical correspondence between bounded cohomology and circle actions developed by Ghys; see [Ghy87]. Also, see [BFH16b] for a thorough treatment of this topic. Let $\mathrm{Homeo}^+(S^1)$ be the group of orientation preserving actions on the circle and let

$$\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) = \{f \in \mathrm{Homeo}^+(\mathbb{R}) \mid \forall n \in \mathbb{Z}, x \in \mathbb{R} : f(x+n) = f(x) + n\}$$

the subgroup of the orientation preserving homeomorphisms of the real line that commutes with the integers. By identifying $S^1 \cong \mathbb{R}/\mathbb{Z}$ we obtain a surjection $\pi\colon \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(S^1)$. The kernel of π is isomorphic to $\mathbb{Z}$ via $\iota\colon n \mapsto f_n$ with $f_n\colon x \mapsto x + n$ and lies in the center of $\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Hence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \iota\ } \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \underset{\curvearrowleft\ \sigma}{\xrightarrow{\ \pi\ }} \mathrm{Homeo}^+(S^1) \longrightarrow 1$$

is a central extension and hence corresponds to a class $\mathrm{eu} \in \mathrm{H}^2(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ the *Euler-class*. This class is represented by the cocycle $\omega\colon (g,h) \mapsto \sigma(g)\sigma(h)\sigma(gh)^{-1} \in \mathbb{Z}$ by identifying $\mathbb{Z}$ with $\ker(\pi) = \mathrm{im}(\iota)$ and where σ is any set-theoretic section $\sigma\colon \mathrm{Homeo}^+(S^1) \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$. Let σ_b be the unique section such that $\sigma_b(f)(0) \in [0,1)$. Then $\omega_b(g,h) = \sigma_b(g)\sigma_b(h)\sigma_b(gh)^{-1}$ satisfies that for all $g, h \in G$, $\omega_b(g,h) \in \{0,1\}$ and hence is ω_b is a *bounded* cocycle. We call the class $\mathrm{eu}_b = [\omega_b] \in \mathrm{H}^2_b(\mathrm{Homeo}^+(S^1), \mathbb{Z})$ the *bounded Euler class*. The image of eu_b under the change of coefficients $\mathrm{H}^2_b(\mathrm{Homeo}^+(S_1), \mathbb{Z}) \to \mathrm{H}^2_b(\mathrm{Homeo}^+(S_1), \mathbb{R})$ is called the *real bounded Euler class* and denoted by $\mathrm{eu}^{\mathbb{R}}_b$.

Any action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ induces a bounded class via $\rho^*\mathrm{eu}_b \in \mathrm{H}^2_b(G, \mathbb{Z})$ (resp. $\rho^*\mathrm{eu}^{\mathbb{R}}_b \in \mathrm{H}^2_b(G, \mathbb{R})$). Ghys ([Ghy87]) showed that two actions $\rho_1, \rho_2\colon G \to \mathrm{Homeo}^+(S_1)$ are *semi-conjugate* if and only if $\rho_1^*\mathrm{eu}_b = \rho_2^*\mathrm{eu}_b \in \mathrm{H}^2_b(G, \mathbb{Z})$. See [BFH16b] for a precise definition of semi-conjugacy. Similarly, we have $\rho^*\mathrm{eu}^{\mathbb{R}}_b = 0 \in \mathrm{H}^2_b(G, \mathbb{R})$ if and only if ρ is semi-conjugate to an action by rotations.

The class $\rho^*\mathrm{eu}_b \in \mathrm{H}^2_b(G, \mathbb{Z})$ may be represented by a cocycle $\rho^*\omega_b \in Z^2_b(G, \mathbb{Z})$ such that for every $g, h \in G$, $\rho^*\omega_b(g,h) \in \{0,1\}$. Surprisingly, a converse statement holds:

Theorem 2.2.4. [1] *Let G be a discrete countable group and let $[\omega] \in H^2_b(G, \mathbb{Z})$ be a class represented by a cocycle ω, such that for all $g, h \in G$, $\omega(g, h) \in \{0, 1\}$. Then there is an action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ such that $\rho^*\mathrm{eu}_b = [\omega] \in \mathrm{H}^2_b(G, \mathbb{Z})$.*

This allows one to show that certain quasimorphisms are induced by a circle action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ without explicitly constructing ρ.

2.2.5 Quasimorphisms

Recall that exact classes $\omega \in \mathrm{H}^2_b(G, \mathbb{R})$ are those classes which vanish in ordinary group cohomology $\mathrm{H}^2(G, \mathbb{R})$. Hence, there is a map $\phi\colon G \to \mathbb{R}$ such that $\delta^1\phi = \omega$. We call such ϕ a *quasimorphism*. To be precise, a quasimorphism is a map $\alpha\colon G \to \mathbb{R}$ such that there is a constant $D > 0$ such that for every $g, h \in G$, $|\alpha(g) - \alpha(gh) + \alpha(h)| \leq D$ and hence $\delta^1\alpha \in C^2_b(G, \mathbb{R})$. The smallest such D is called the *defect* of α and denoted by $D(\alpha)$.

A quasimorphism $\alpha\colon G \to \mathbb{R}$ will be called *symmetric* if α satisfies in addition that $\alpha(g) = -\alpha(g^{-1})$ for all $g \in G$. It is easy to see that each exact 2-class is represented by a symmetric cocycle.

A quasimorphism $\bar{\phi}$ is said to be *homogeneous* if $\bar{\phi}(g^n) = n\bar{\phi}(g)$ for all $n \in \mathbb{Z}$, $g \in G$. In particular, $\bar{\phi}$ is *symmetric*, i.e. $\bar{\phi}(g^{-1}) = -\bar{\phi}(g)$ for all $g \in G$.

Every quasimorphism $\phi\colon G \to \mathbb{R}$ is boundedly close to a unique homogeneous quasimorphism $\bar{\phi}\colon G \to \mathbb{R}$ defined via

$$\bar{\phi}(g) := \lim_{n \to \infty} \frac{\phi(g^n)}{n}$$

and we call $\bar{\phi}$ the *homogenisation* of ϕ. Homogeneous quasimorphisms on G form a vector space, denoted by $\mathcal{Q}(G)$.

Proposition 2.2.5 ([Cal09b, Lemma 2.58])**.** *Let $\phi\colon G \to \mathbb{R}$ be a quasimorphism and let be $\bar{\phi}$ be its homogenisation. Then $D(\bar{\phi}) \leq 2D(\phi)$.*

We will decorate homogeneous quasimorphisms with a bar-symbol, even if they are not explicitly induced by a non-homogeneous quasimorphism.

On a non-abelian free group F there are several constructions of non-trivial quasimorphisms.

[1] See [Ghy87], see also Theorem 1.3 of [BFH16b]

Example 2.2.6. In [Bro81], Brooks gave the first example of an infinite family of linearly independent quasimorphisms on the free group. Let F be a non-abelian free group on a fixed generating set $\mathcal{S}$. Let $w, g \in F$ be two elements which are represented by reduced words $w = \mathtt{y}_1 \cdots \mathtt{y}_n$ and $g = \mathtt{x}_1 \cdots \mathtt{x}_m$, where $\mathtt{x}_j, \mathtt{y}_j$ are letters of F. We say that w is a sub-word of g if $n \leq m$ and there is an $s \in \{0, \dots, m-n\}$ such that $\mathtt{y}_j = \mathtt{x}_{j+s}$ for all $j \in \{1, \dots, n\}$. Let w be a reduced *non self-overlapping* word, i.e. a word w such that there are no words x and y with x non-trivial such that $w = xyx$ as a reduced word. For w a non self-overlapping word we define the function $\nu_w \colon F \to \mathbb{Z}$ by setting $\nu_w \colon g \mapsto \#\{w \text{ is a subword of } g\}$. Then the *Brooks counting quasimorphism on the word w* is the function

$$\phi_w = \nu_w - \nu_{w^{-1}}.$$

It is easy to see that this defines a symmetric quasimorphism.

Example 2.2.7. In [Rol09], Rolli gave a different example of an infinite family of linearly independent quasimorphisms. Suppose F is generated by $\mathcal{S} = \{\mathtt{x}_1, \dots, \mathtt{x}_n\}$. Let $\lambda_1, \dots, \lambda_n \in \ell^\infty_{alt}(\mathbb{Z})$ be bounded functions $\lambda_j \colon \mathbb{Z} \to \mathbb{R}$ that satisfy $\lambda_j(-n) = -\lambda_j(n)$. Each non-trivial element $g \in F$ may be uniquely written as $g = \mathtt{x}_{n_1}^{m_1} \cdots \mathtt{x}_{n_k}^{m_k}$ where all m_j are non-zero and no consecutive n_j are the same. Then we can see that the map $\phi \colon F \to \mathbb{R}$ defined by setting

$$\phi \colon g \mapsto \sum_{j=1}^{k} \lambda_{n_j}(m_j)$$

is a symmetric quasimorphism called *Rolli quasimorphism*. We can estimate the defect by $\max\{3\|\lambda_i\|\}$.

2.2.6 Generalised Quasimorphisms

It is possible to generalise quasimorphisms $\phi \colon G \to \mathbb{R}$ to maps $\Phi \colon G \to H$ for G, H *arbitrary groups*. Two quite different proposals for such a generalisation come from Fujiwara–Kapovich ([FK16]) and Hartnick–Schweitzer ([HS16]). Whereas the former maps are quite restrictive, the latter type of maps are very rich.

2.2.6.1 Quasihomomorphisms by Fujiwara–Kapovich

Ordinary quasimorphisms $\phi\colon G \to \mathbb{Z}$ may be characterized by their "bounded defect". In [FK16], Fujiwara and Kapovich adopted this notion for general groups:

Definition 2.2.8. [2] Let G and H be groups and let $\sigma\colon G \to H$ be a set-theoretic map. Define $\mathrm{d}\colon G \times G \to H$ via $\mathrm{d}\colon (g,h) \mapsto \sigma(g)\sigma(h)\sigma(gh)^{-1}$ and define $D(\sigma) \subset H$, the *defect of* σ via

$$D(\sigma) = \{\mathrm{d}(g,h) | g,h \in G\} = \{\sigma(g)\sigma(h)\sigma(gh)^{-1} \mid g,h \in G\}.$$

The group $\Delta(\sigma) < H$ generated by $D(\sigma)$ is called the *defect group*. The map $\sigma\colon G \to H$ is called *quasihomomorphism* if the defect $D(\sigma) \subset H$ is finite.

When there is no danger of ambiguity we will write $D = D(\sigma)$ and $\Delta = \Delta(\sigma)$. This definition is slightly different from the original definition in [FK16]. Here, the authors required that the set

$$\bar{D}(\sigma) = \{\sigma(h)^{-1}\sigma(g)^{-1}\sigma(gh) \mid g,h \in G\}$$

is finite. However, those two definitions may be seen to be equivalent:

Proposition. *Let G, H be groups and let $\sigma\colon G \to H$ be a set-theoretic map. Then σ is a quasihomomorphism in the sense of Definition 2.2.8 if and only if it is a quasihomomorphism in the sense of Fujiwara–Kapovich ([FK16]) i.e. if and only if $\bar{D}(\sigma)$ is finite.*

This will be shown in Proposition 3.1.6. Every set theoretic map $\sigma\colon G \to H$ with finite image and every homomorphism is a quasihomomorphisms for "trivial" reasons. We may also construct different quasihomomorphisms using quasimorphisms $\phi\colon G \to \mathbb{Z}$: Let $C < H$ be an infinite cyclic subgroup and let $\tau\colon \mathbb{Z} \to H$ be a homomorphism s.t. $\tau(\mathbb{Z}) = C$. Then it is easy to check that for every quasimorphism $\phi\colon G \to \mathbb{Z}$, $\tau \circ \phi\colon G \to H$ is a quasihomomorphism.

Fujiwara–Kapovich showed that if the target H is a torsion-free hyperbolic group then the above mentioned maps are the only possible quasihomomorphisms. To be precise in this case every quasihomomorphism $\sigma\colon G \to H$ has either finite image, is a homomorphism, or maps to a cyclic subgroup of H; see Theorem 4.1 of [FK16].

[2] see [FK16]

2.2.6.2 Quasimorphisms by Hartnick–Schweitzer

In [HS16], Hartnick and Schweitzer proposed a different generalisation of "quasimorphisms". This approach is more functorial.

Definition 2.2.9 ([HS16])**.** a map $\Phi \colon G \to H$ between arbitrary groups a *quasimorphism* if for every (ordinary) quasimorphism $\alpha \colon H \to \mathbb{R}$, $\alpha \circ \Phi \colon G \to \mathbb{R}$, i.e. the pullback of α to G via Φ, defines a quasimorphism on G.

Note that a map $\phi \colon G \to \mathbb{R}$ is a quasimorphism in the sense of Hartnick–Schweitzer if and only if it is an ordinary quasimorphism. There are many more quasimorphisms between groups as there are quasihomomorphisms. In chapter 5 we will construct quasimorphisms between free groups.

2.3 Stable Commutator Length and Bavard's Duality Theorem

Let G be a group. For two elements $g, h \in G$ the *commutator* is defined via $[g,h] = ghg^{-1}h^{-1}$ and the group generated by all such commutators is called the *commutator subgroup* of G and is denoted by G'. For an element $g \in G'$ we set

$$\mathrm{cl}(g) = \min\{k \mid g = \prod_{i=1}^{k} [g_i, h_i]; g_i, h_i \in G\}$$

the *commutator length of* g. Note that cl is subadditive and hence the limit

$$\mathrm{scl}(g) = \lim_{n \to \infty} \frac{\mathrm{cl}(g^n)}{n}$$

exists and is called *stable commutator length (scl)*. See [Cal09b] for a comprehensive reference on scl. Calegari showed that in non-abelian free groups scl can be computed efficiently in polynomial time and is rational. For a group G, the set of possible values of scl is not fully understood, even for non-abelian free groups. Stable commutator length has the following geometric meaning. Let X be a topological space with $\pi_1(X) = G$ and let $\gamma \colon S^1 \to X$ be a loop representing the conjugacy class of an element $[\gamma] \in G'$ in the commutator subgroup of G. A map

$\phi\colon \Sigma \to X$ from a surface Σ to X is called *admissible* if there is a commutative diagram

$$\begin{array}{ccc} \partial\Sigma & \xrightarrow{\iota} & \Sigma \\ \big\downarrow{\scriptstyle \partial\phi} & & \big\downarrow{\scriptstyle \phi} \\ S^1 & \xrightarrow{\gamma} & X \end{array}$$

where $\iota\colon \partial\Sigma \to \Sigma$ is the inclusion map of the boundary of Σ. Define $n(\Sigma, \phi)$ by setting $\partial\phi[\partial\Sigma] = n(\Sigma, \phi)[S^1]$ in $\mathrm{H}_1(S^1)$. Further, define $\chi^-(\Sigma)$ via $\chi^-(\Sigma) = \min\{\chi(\Sigma), 0\}$, where $\chi(\Sigma)$ is the Euler characteristic of Σ. Calegari showed that the stable commutator length of $[\gamma] \in G'$ is an obstruction to how efficiently γ may be bounded by surfaces Σ:

Theorem ([Cal09b])**.** *Let X, G and γ be as above. Then*

$$\mathrm{scl}([\gamma]) = \inf_{(\Sigma,\phi)} \frac{-\chi^-(\Sigma)}{2n(\Sigma, \phi)}$$

where the infimum is taken over all admissible pairs (Σ, ϕ).

We observe the following basic property:

Proposition 2.3.1. scl *is monotone and characteristic. That is, for any group homomorphism $\theta\colon G \to H$ and any $g \in G$ we have* $\mathrm{scl}(g) \geq \mathrm{scl}(\theta(g))$. *If θ is an automorphism, then* $\mathrm{scl}(g) = \mathrm{scl}(\theta(g))$.

There is an intriguing connection between stable commutator length and bounded cohomology via quasimorphisms.

Theorem 2.3.2 ([Bav91])**.** *Let G be a group and let $g \in G'$. Then*

$$\mathrm{scl}(g) = \sup_{\bar{\phi}} \frac{|\bar{\phi}(g)|}{2D(\bar{\phi})}$$

where the supremum is taken over all homogeneous quasimorphisms $\bar{\phi}\colon G \to \mathbb{R}$.

See [Cal09b] for a proof and a generalisation of this statement. This theorem allows us to estimate stable commutator length using (homogeneous) quasimorphisms. It can be shown that the supremum in Bavard's Duality Theorem is obtained. That is, for every element $g \in G'$ there is a homogeneous quasimorphism

$\bar{\phi}$ with $D(\bar{\phi}) = 1$ such that $\mathrm{scl}(g) = \bar{\phi}(g)/2$. These quasimorphisms are called *extremal* and were studied in [Cal09a]. Extremal quasimorphisms are usually hard to construct.

Example 2.3.3 (Free Groups)**.** In free groups stable commutator length can be computed in polynomial time using an algorithm of Calegari [Cal09b]. His algorithm further showed that stable commutator length is rational in free groups and revealed a surprising distribution of stable commutator length.

This algorithm computes the infimum of the euler characteristic of surfaces mapping to a space X with free fundamental group. Only in few cases are the extremal quasimorphisms to elements in non-abelian free groups explicitly known.

Let $F = \langle \mathtt{a}, \mathtt{b} \rangle$ be the group freely generated by the letters $\mathtt{a}, \mathtt{b}$ and let ϕ_w be the Brooks quasimorphism on the letter w as described in Example 2.2.6. Consider $[\mathtt{a}, \mathtt{b}]$, the commutator of the letters $\mathtt{a}$ and $\mathtt{b}$. Then it is easy to see that the quasimorphism $\eta_0 = \phi_{\mathtt{ab}} - \phi_{\mathtt{ba}}$ satisfies that $\eta_0([\mathtt{a}, \mathtt{b}]) = \bar{\eta}_0([\mathtt{a}, \mathtt{b}]) = 2$, $D(\eta_0) = 1$ and $D(\bar{\eta}_0) = 2$. As usual, $\bar{\eta}_0$ denotes the homogenisation of η_0. By Bavard's Duality Theorem (2.3.2) we may estimate $\mathrm{scl}([\mathtt{a}, \mathtt{b}]) \geq \bar{\eta}([\mathtt{a}, \mathtt{b}])/2D(\bar{\eta}) = 1/2$ and, as $\mathrm{scl}([\mathtt{a}, \mathtt{b}]) \leq 1/2$ (see Section 2.3), we conclude $\mathrm{scl}([\mathtt{a}, \mathtt{b}]) = 1/2$ and see that $\bar{\eta}_0$ is extremal.

We will generalise this example to construct quasimorphisms for arbitrary commutators in free non-abelian groups in Chapter 5.

2.3.1 Vanishing of stable commutator length

An element $g \in G'$ may satisfy that $\mathrm{scl}(g) = 0$ for "trivial" reasons, such as if g is torsion or if g is conjugate to its inverse. There are many classes of groups where – besides these trivial reasons – stable commutator length vanishes on the whole group.

By Bavard's Duality principle this is equaivalent to the injectivity of the comparison map $c^2_G \colon H^2_b(G; \mathbb{R}) \to H^2(G; \mathbb{R})$. Examples include:

- amenable groups: This follows from the vanishing of $H^2_b(G; \mathbb{R})$ for every amenable group G by a result of Trauber [Gro82],
- irreducible lattices in semisimple Lie groups of rank at least 2 [BM02], and

- subgroups of the group $\mathrm{PL}^+(I)$ of piecewise linear transformations of the interval [Cal07].

2.3.2 Gaps in stable commutator length

It was shown by [DH91] that every non-trivial element $w \in F'$ in the commutator subgroup of a non-abelian free group F satisfies that $\mathrm{scl}(w) \geq 1/2$ and that equality $\mathrm{scl}(w) = 1/2$ if w is a commutator.

Using the monotonicity of scl we may conclude that for an arbitrary group G every commutator $[g_1, g_2] \in G'$ satisfies $\mathrm{scl}([g_1, g_2]) \leq 1/2$. On the other hand, some elements $g \in G'$ satisfy $\mathrm{scl}(g) = 0$ for trivial reasons, for example if they are torsion or a positive power of this element is conjugate to a negative power of this element.

We call the infimum of $\{scl(g) > 0 \mid g \in G'\}$ the *gap of* scl, often called the *spectral gap*, and say that a group *has a gap in scl* if this number is positive. Many classes of "negatively curved" groups have a gap in scl.

- Residually free groups have a gap of exactly 1/2 by Duncan and Howie [DH91].

- Mapping class groups of closed orientable surfaces, possibly with punctures, have a gap depending on the surface; see [BBF16].

- Hyperbolic groups have a gap which depends on the hyperbolicity constant and the number of generators; see [CF10].

- Some classes of groups may not have a uniform gap but the first accumulation point on conjugacy classes of positive scl may be uniformly bounded away from zero. For example for non-elementary, torsion free hyperbolic groups and for the fundamental groups of closed hyperbolic manifolds this accumulation point is at least 1/12; see Theorem B of [CF10] and see Theorem 3.11 of [Cal09b].

- Sometimes, one may control scl on certain generic group elements. If $G = G_1 \star G_2$ is the free product of two torsion free groups G_1 and G_2 and $g \in G'$ does not conjugate into one of the factors, then $\mathrm{scl}(g) \geq 1/2$; see [Che18] and

[IK17]. Similarly, if $G = A \star_C B$ and $g \in G'$ does not conjugate into one of the factors and such that CgC does not contain a copy of any conjugate of g^{-1} then $\mathrm{scl}(g) \geq 1/12$. See Theorem D of [CF10] for the first proof of this gap and [CFL16] for the sharp gap and a generalisation to graphs of groups.

- Baumslag–Solitar groups have a sharp uniform gap of $1/12$; see [CFL16].

Note that this list is not meant to be comprehensive. By monotinicity, having a gap in scl may serve as an obstruction for group embeddings. If H and G are non-abelian groups with $H \hookrightarrow G$ and $C > 0$ is such that every non-trivial element $g \in G'$ satisfies $\mathrm{scl}(g) \geq C$ then so does every non-trivial element of H'.

2.4 Simplicial volume and l^1-semi norms

An important application of bounded cohomology is to study and computation of simplicial volume of manifolds.

We recall the l^1-semi-norm on real homology and simplicial volume in Subsection 2.4.1. In Subsection 2.4.2 we recall describe the set of simplicial volumes in dimensions 2 and 3 and discuss known examples of simplicial volumes in dimensions 4. In Subsection 2.4.3 we state well known properties of simplicial volume. In Subsection 2.4.4 we discuss the relationship between simplicial volume and bounded cohomology.

2.4.1 The l^1-semi-norm and simplicial volume

Let X be a topological space, let $n \in \mathbb{N}$ be an integer and let $\alpha \in \mathrm{H}(X, \mathbb{R})$ be a class. The *l^1-semi-norm* $\|\alpha\|_1$ of α is defined as

$$\|\alpha\|_1 = \inf\big\{ |c|_1 \bigm| c \in C_d(X, \mathbb{R}),\ \partial c = 0,\ [c] = \alpha \big\},$$

where $C_d(X, \mathbb{R})$ is the singular chain module of X in degree n with $\mathbb{R}$-coefficients and $|\cdot|_1$ denotes the l^1-norm on $C_d(X, \mathbb{R})$ associated with the basis of singular simplices.

Definition 2.4.1 ([Gro82])**.** Let M be an oriented closed connected d-dimensional manifold. Then the *simplicial volume of M* is defined by

$$\|M\| := \big\|[M]_{\mathbb{R}}\big\|_1,$$

where $[M]_{\mathbb{R}} \in H_d(M;\mathbb{R})$ denotes the $\mathbb{R}$-fundamental class of M.

Simplicial volume is independent of the sign of the fundamental class hence we will talk about the simplicial volume of orientable manifolds.

2.4.2 Simplicial volume in low dimensions and gaps

For an integer $d \geq 2$ we define $\mathrm{SV}(d) \subset \mathbb{R}_{\geq 0}$ the set of simplicial volumes of orientable closed connected d-manifolds via

$$\mathrm{SV}(d) := \big\{\|M\| \;\big|\; M \text{ is an orientable closed connected } d\text{-manifold}\big\}.$$

There are only countably many homotopy types of orientable closed connected manifolds. Hence the set $\mathrm{SV}(d)$ is countable for every $d \in \mathbb{N}$.

The set $\mathrm{SV}(d)$ is also closed under addition. For $d \geq 3$, this follows from the additivity of simplicial volume under connected sums [Gro82][Fri17, Corollary 7.7] and for $d = 2$ this follows from the explicit computation of $\mathrm{SV}(2)$ as seen in Example 2.4.2.

Example 2.4.2 (dimension 2)**.** For an orientable closed surface Σ_g of genus $g \geq 1$ we have $\|\Sigma_g\| = 2 \cdot \big|\chi(\Sigma_g)\big| = 4 \cdot (g-1)$ [Gro82][Fri17, Corollary 7.5]. Hence,

$$\mathrm{SV}(2) = \{0, 4, 8, \ldots\} = \mathbb{N}[4].$$

We observe that the *gap* in simplicial volume of dimension 2 is 4.

Example 2.4.3 (dimension 3)**.** We have [Gro82][Fri17, Corollary 7.8]

$$\mathrm{SV}(3) = \mathbb{N}\Big[\frac{\mathrm{vol}(M)}{v_3} \;\Big|\; M \text{ is a complete hyperbolic 3-manifold with toroidal boundary and finite volume}\Big]$$

and where v_3 is the maximum volume of an ideal simplex in $\mathbb{H}^3$. This shows that there is a gap of simplicial volume in dimension 3, namely $w/v_3 \approx 0.928\ldots$, where w is the volume of the Weeks manifold [GMM09].

Example 2.4.4 (dimension 4)**.** The smallest known Riemannian volume vol(M) of an orientable closed connected *hyperbolic* 4-manifold is $64 \cdot \pi^2/3$ [CM05]. In view of the computation of the simplicial volume of hyperbolic manifolds [Gro82][Fri17, Chapter 7.3] this means that the smallest known simplicial volume of a hyperbolic oriented closed connected 4-manifold is $\frac{64 \cdot \pi^2}{3 \cdot v_4} \in [700, 800]$ where v_4 is the maximum volume of an ideal simplex in $\mathbb{H}^4$.

If Σ_g, Σ_h are orientable closed connected surfaces of genus $g, h \geq 1$, respectively, then Bucher [BK08] showed that $\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \cdot \|\Sigma_g\| \cdot \|\Sigma_h\|$. Hence, $\|\Sigma_2 \times \Sigma_2\| = 24$. This has been the smallest known non-trival simplicial volume of a 4-manifold.

2.4.3 Properties of simplicial volume

2.4.3.1 Minimal Volume

Define the *minimal volume* minVol(M) of a closed manifold M as the infimum of the volumes of Riemannian metrics supported on M with sectional curvature between -1 and 1; see [Fri17, Chapter 7.8].

Gromov showed that simplicial volume (suitably normalised) bounds the *minimal volume* of Riemannian Manifolds, namely that for a closed n-manifolds

$$\frac{\|M\|}{(n-1)^n n!} \leq \text{minVol}(M)$$

holds.

2.4.3.2 Products and connected sums

For orientable closed connected manifolds M and N of dimension m and n respectively we can estimate the cross product $M \times N$ as

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{n+m}{m} \cdot \|M\| \cdot \|N\|.$$

If M and N are both orientable closed connected n-manifolds for $n \geq 3$ we have

$$\|M \# N\| = \|M\| + \|N\|$$

where $M \# N$ denotes the connected sum of M and N. A similar statement holds for glueings with amenable boundaries. These results can be found in Chapter 7 of [Fri17].

2.4.4 Duality

Bounded cohomology of groups and spaces may be used to compute the l^1-semi-norm of homology classes and hence of simplicial volume. For what follows, let $\langle\cdot,\cdot\rangle\colon H_b^n(X;V)\times H_n(X;V)\to V$ be the map given by evaluation of cochains on chains.

Proposition 2.4.5 (duality principle [Fri17, Lemma 6.1])**.** *Let X be a topological space and let $\alpha\in H_n(X;\mathbb{R})$. Then*

$$\|\alpha\|_1=\sup\bigl\{\langle\beta,\alpha\rangle \bigm| \beta\in \mathrm{H}_b^n(X,\mathbb{R}),\ \|\beta\|_\infty\le 1\bigr\}.$$

Moreover, the supremum is achieved.

Cocycles $\beta\in C_b^n(X,\mathbb{R})$ that satisfy $\|\beta\|_\infty=1$ and $\langle[\beta],\alpha\rangle=\|\alpha\|_1$ are called *extremal for* α.

Corollary 2.4.6 (mapping theorem for the l^1-semi-norm)**.** *Let $f\colon X\longrightarrow Y$ be a continuous map between path-connected topological spaces. If the induced homomorphism $\pi_1(f)\colon\pi_1(X)\longrightarrow\pi_1(Y)$ is surjective and has amenable kernel, then $H_\bullet(f,\mathbb{R})\colon H_\bullet(X,\mathbb{R})\longrightarrow H_\bullet(Y,\mathbb{R})$ is isometric with respect to the l^1-semi-norm.*

Proof. We only need to combine the duality principle (Proposition 2.4.5) with the mapping theorem in bounded cohomology (Theorem 2.2.3). $\square$

Chapter 3

Group Extensions and Bounded Cohomology

The material in this chapter is taken from [Heu17b]. The bounded cohomology of a group G with trivial real coefficients is notoriously hard to compute: There is no full characterisation of all bounded classes in $\mathrm{H}_b^n(G, \mathbb{R})$ for $n = 2, 3$. For $n \geq 4$, $\mathrm{H}_b^n(G, \mathbb{R})$ is usually fully unknown, even if G is a non-abelian free group.

On the other hand, for *ordinary* n-dimensional group cohomology $\mathrm{H}^n(G, V)$ there is a well-known characterisation for $n = 2, 3$ in terms of *group extensions*. The aim of this chapter is to develop an analogous correspondence for *bounded* cohomology. For this, we first recall the classical connection between group extensions and ordinary group cohomology.

Definition 3.0.1. An *extension* of a group G by a group N is a short exact sequence of groups

$$1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1. \tag{3.1}$$

We say that two group extensions $1 \to N \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} G \to 1$ and $1 \to N \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} G \to 1$ of G by N are *equivalent*, if there is an isomorphism $\Phi\colon E_1 \to E_2$ such that the diagram

$$\begin{array}{ccccccc} & & & E_1 & & & \\ & & \overset{\iota_1}{\nearrow} & & \overset{\pi_1}{\searrow} & & \\ 1 \longrightarrow & N & & \Big\downarrow \Phi & & G & \longrightarrow 1 \\ & & \underset{\iota_2}{\searrow} & & \underset{\pi_2}{\nearrow} & & \\ & & & E_2 & & & \end{array}$$

commutes.

Any group extension of G by N induces a homomorphism $\psi\colon G \to \mathrm{Out}(N)$; see Subsection 3.2.1. Two equivalent extensions of G by N induce the same such map $\psi\colon G \to \mathrm{Out}(N)$. We denote by $\mathcal{E}(G, N, \psi)$ the set of group extensions of G by N which induce ψ under this equivalence. If there is no danger of ambiguity we do not label the maps of the short exact sequence i.e. we will write $1 \to N \to E \to G \to 1$ instead of (3.1).

It is well-known that one may fully characterise $\mathcal{E}(G, N, \psi)$ in terms of ordinary group cohomology:

Theorem 3.0.2 ([Bro82, Theorem 6.6] [Mac49])**.** *Let G and N be groups and let $\psi\colon G \to \mathrm{Out}(N)$ be a homomorphism. Furthermore, let $Z = Z(N)$ be the centre of N equipped with the action of G induced by ψ. Then there is a class $\omega = \omega(G, N, \psi) \in \mathrm{H}^3(G, Z)$, called* obstruction*, such that $\omega = 0$ in $\mathrm{H}^3(G, Z)$ if and only if $\mathcal{E}(G, N, \psi) \neq \emptyset$. In this case there is a bijection between the sets $\mathrm{H}^2(G, Z)$ and $\mathcal{E}(G, N, \psi)$.*

Moreover, for a G-module Z it is possible to characterise $\mathrm{H}^3(G, Z)$ in terms of these obstructions:

Theorem 3.0.3 ([Bro82, Section IV, 6])**.** *For any G-module Z and any $\alpha \in \mathrm{H}^3(G, Z)$ there is a group N with $Z = Z(N)$ and a homomorphism $\psi\colon G \to \mathrm{Out}(N)$ extending the action of G on Z such that $\alpha = \omega(G, N, \psi)$.*

In other words, any three dimensional class in ordinary cohomology arises as an obstruction.

The aim of this chapter is to derive analogous statements to Theorem 3.0.2 and Theorem 3.0.3 involving *bounded* cohomology. This will use *quasihomomorphisms* as defined and studied by Fujiwara–Kapovich in [FK16]. Let G and H be groups. A set-theoretic function $\sigma\colon G \to H$ is called *quasihomomorphism* if the set

$$D(\sigma) = \{\sigma(g)\sigma(h)\sigma(gh)^{-1} | g, h \in G\}$$

is finite. We note that this is not the original definition of [FK16] but both definitions are equivalent; see Proposition 3.1.6.

Definition 3.0.4. We say that an extension $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ of G by N is *bounded*, if there is a (set theoretic) section $\sigma\colon G \to E$ such that

(i) $\sigma\colon G \to E$ is a quasihomomorphism and

(ii) the map $\phi_\sigma\colon G \to Aut(N)$ induced by σ has finite image in $\mathrm{Aut}(N)$.

Here $\phi_\sigma\colon G \to \mathrm{Aut}(N)$ denotes the set-theoretic map $\phi_\sigma\colon g \mapsto \phi_\sigma(g)$ with

$$^{\phi_\sigma(g)}n = \iota^{-1}(\sigma(g)\iota(n)\sigma(g)^{-1}).$$

We stress that ϕ_σ is in general not a homomorphism. See Remark 3.1.2 for the notation. Condition *(ii)* may seem artificial but is both natural and necessary; see Remark 3.1.4. We denote the set of all bounded extensions of a group G by N which induce ψ by $\mathcal{E}_b(G, N, \psi)$ and mention that this is a subset of $\mathcal{E}(G, N, \psi)$.

Analogously to Theorem 3.0.2 we will characterise the set $\mathcal{E}_b(G, N, \psi) \subset \mathcal{E}(G, N, \psi)$ using *bounded* cohomology.

Theorem A. *Let G and N be groups and suppose that $Z = Z(N)$, the centre of N, is equipped with a norm $\|\cdot\|$ such that $(Z, \|\cdot\|)$ has finite balls. Furthermore, let $\psi\colon G \to \mathrm{Out}(N)$ be a homomorphism with finite image.*

There is a class $\omega_b = \omega_b(G, N, \psi) \in \mathrm{H}^3_b(G, Z)$ such that $\omega_b = 0$ in $\mathrm{H}^3_b(G, Z)$ if and only if $\mathcal{E}_b(G, N, \psi) \neq \emptyset$ and $c^3(\omega_b) = \omega$ is the obstruction of Theorem 3.0.2. If $\mathcal{E}_b(G, N, \psi) \neq \emptyset$, then the bijection between the sets $\mathrm{H}^2(G, Z)$ and $\mathcal{E}(G, N, \psi)$ described in Theorem 3.0.2 restricts to a bijection between $im(c^2) \subset \mathrm{H}^2(G, Z)$ and $\mathcal{E}_b(G, N, \psi) \subset \mathcal{E}(G, N, \psi)$.

Here, $c^n\colon \mathrm{H}^n_b(G, Z) \to \mathrm{H}^n(G, Z)$ denotes the *comparison map*; see Subsection 2.2.1. We say that a normed group or module $(Z, \|\cdot\|)$ has finite balls if for every $K > 0$ the set $\{z \in Z \mid \|z\| \leq K\}$ is finite. Theorem A is applied to examples in Subsection 3.4.1.

Just as in Theorem 3.0.3 we may ask which elements of $\mathrm{H}^3_b(G, Z)$ may be realised by obstructions. For a G-module Z we define the following subset of $\mathrm{H}^3_b(G, Z)$:

$$\mathcal{F}(G, Z) := \{\Phi^*\alpha \in \mathrm{H}^3_b(G, Z) \mid \Phi\colon G \to M, \alpha \in \mathrm{H}^3_b(M, Z)\}$$

where $\Phi: G \to M$ is a homomorphism, M is a finite group, and where $\Phi^*\alpha$ denotes the pullback of α via the homomorphism Φ. Note that as M is finite, $\mathrm{H}^3(M, Z) = \mathrm{H}^3_b(M, Z)$. Analogously to Theorem 3.0.3 we will show:

Theorem B. *Let G be a group, let Z be a normed G-module with finite balls and such that G acts on Z via finitely many automorphisms. Then*

$$\{\omega_b(G, N, \psi) \in \mathrm{H}^3_b(G, Z) \mid Z = Z(N) \text{ and } \psi \text{ induces the action on } G\} = \mathcal{F}(G, Z)$$

as subsets of $\mathrm{H}^3_b(G, Z)$.

As finite groups are amenable this shows that all such classes in $\mathrm{H}^3_b(G, Z)$ will vanish under a change to real coefficients; see Subsection 2.2.1. We prove Theorem A and B following the outline of the classical proofs in [Bro82].

Organisation of this chapter

This chapter is organised as follows: In Section 3.1 we recall well-known facts about outer automorphisms and prove the equivalent definitions of quasihomomorphisms. In Section 3.2 we will reformulate the problem of characterising group extensions using *non-abelian cocycles*; see Definition 3.2.2. Using this characterisation, we will prove Theorem A in Subsection 3.2.4. In Section 3.3 we prove Theorem B which characterises the set of classes arising as obstructions ω_b. In Section 3.4 we give examples to show that the assumptions of Theorem A are necessary and discuss generalisations.

3.1 Preliminaries

In this section we recall notation and conventions regarding the (outer) automorphisms in Subsection 3.1.1 and recall and discuss quasihomomorphisms in Subsection 3.1.3.

3.1.1 Aut and Out

Let N be a group and let $\mathrm{Aut}(N)$ be the group of automorphisms of N. Recall that $\mathrm{Inn}(N)$ denotes the group of *inner automorphisms*. This is, the subgroup of

$\mathrm{Aut}(N)$ whose elements are induced by conjugations of elements in N. There is a map $\phi\colon N \to \mathrm{Inn}(N)$ via $\phi\colon n \to \phi_n$ where $\phi_n\colon g \mapsto ngn^{-1}$. Recall that $\mathrm{Inn}(N)$ is a normal subgroup of $\mathrm{Aut}(N)$ and that the quotient $\mathrm{Out}(N) = \mathrm{Aut}(N)/\mathrm{Inn}(N)$ is the *group of outer automorphisms of* N. It is well-known that there is an exact sequence

$$1 \to Z \to N \to \mathrm{Inn}(N) \to \mathrm{Aut}(N) \to \mathrm{Out}(N) \to 1$$

where $Z = Z(N)$ denotes the centre of N and all the maps are the natural ones. We will frequently use the following facts. Let G be a group. Any homomorphism $\psi\colon G \to \mathrm{Out}(N)$ induces an action on $Z = Z(N)$. This fact is also proved in detail in Subsection 3.2.1. Moreover, if $n_1, n_2 \in N$ are two elements such that for every $g \in N$, $\phi_{n_1}(g) = \phi_{n_2}(g)$ then n_1 and n_2 just differ by an element in the centre, i.e. there is $z \in Z(N)$ such that $n_1 = zn_2$. This may be seen by the exactness of the above sequence.

3.1.2 Non-degenerate cocycles

To show Theorem B of this chapter it will be helpful to work with *non-degenerate* chains. A map $\alpha \in C^n(G,V)$ is called non-degenerate if $\alpha(g_1,\dots,g_n) = 0$ whenever $g_i = 1$ for some $i = 1,\dots,n$. We define $NC^0(G,V) = NC_b^0(G,V) = V$ and moreover $NC^n(G,V) = \{\alpha \in C^n(G,V) \mid \alpha \text{ non-degenerate}\}$ and $NC_b^n(G,V) = \{\alpha \in C_b^n(G,V) \mid \alpha \text{ non-degenerate}\}$ and observe that δ^* sends non-degenerate maps to non-degenerate maps.

Proposition 3.1.1. *The homology of* $(NC^*(G,V),\delta^*)$ *is* $\mathrm{H}^n(G,V)$ *and the homology of* $(NC_b^*(G,V),\delta^*)$ *is* $\mathrm{H}_b^n(G,V)$.

Proof. See Section 6 of [Mac67], where an explicit homotopy between the complexes $(NC^*(G,V),\delta^*)$ and $(C^*(G,V),\delta^*)$ is constructed. Moreover, one may see that this homotopy preserves bounded maps and hence yields a homotopy between $(NC_b^*(G,V),\delta^*)$ and $(C_b^*(G,V),\delta^*)$. □

3.1.3 Properties of quasihomomorphisms

Recall from Subsection 2.2.6 that a (set-theoretic) map $\sigma\colon G \to H$ between two groups G and H is called *quasihomomorphisms* if the set

$$D(\sigma) = \{\sigma(g)\sigma(h)\sigma(gh)^{-1} \mid g, h \in G\}$$

is finite; see Definition 2.2.8. This set is called the *defect set* and the group $\Delta(\sigma) < H$ generated by $D(\sigma)$ is called the *defect group*. The map $\mathrm{d}\colon G \times G \to D(\Sigma) < H$ defined by

$$\mathrm{d}\colon (g, h) \mapsto \sigma(g)\sigma(h)\sigma(gh)^{-1}$$

is called the *defect map*.

This definition is slightly different from the definition of Fujiwara–Kapovich in [FK16]. Here, the authors called a map $\sigma\colon G \to H$ a quasihomomorphism if the set

$$\bar{D}(\sigma) = \{\sigma(h)^{-1}\sigma(g)^{-1}\sigma(gh) \mid g, h \in G\}$$

is finite. We show that both definitions agree in Proposition 3.1.6, proven at the end of this Subsection.

Every set theoretic map $\sigma\colon G \to H$ with finite image and every homomorphism are quasihomomorphisms for "trivial" reasons. We may also construct different quasihomomorphisms using quasimorphisms $\phi\colon G \to \mathbb{Z}$: Let $C < H$ be an infinite cyclic subgroup and let $\tau\colon \mathbb{Z} \to H$ be a homomorphism such that $\tau(\mathbb{Z}) = C$. Then it is easy to check that for every quasimorphism $\phi\colon G \to \mathbb{Z}$ the map $\tau \circ \phi\colon G \to H$ is a quasihomomorphism.

Fujiwara–Kapovich showed that if the target H is a torsion-free hyperbolic group then the above mentioned maps are the only possible quasihomomorphisms. To be precise in this case every quasihomomorphism $\sigma\colon G \to H$ has either finite image, is a homomorphism, or maps to a cyclic subgroup of H; see Theorem 4.1 of [FK16].

We recall basic properties of quasihomomorphisms. For what follows we use the following convention.

Remark 3.1.2. If $\alpha \in \mathrm{Aut}(G)$ and $g \in G$ then ${}^{\alpha}g$ denotes the element $\alpha(g) \in G$. If $a \in G$ is an element then ${}^{a}g$ denotes conjugation by a, i.e. the element $aga^{-1} \in G$.

Sometimes we successively apply automorphisms and conjugations. For example, ${}^{a\alpha}g$ denotes the element $a\alpha(g)a^{-1} \in G$.

Proposition 3.1.3 ([FK16, Lemma 2.5])**.** *Let $\sigma\colon G \to H$ be a quasihomomorphism, let D and Δ be as above and let $H_0 < H$ be the subgroup of H generated by $\sigma(G)$. Then Δ is normal in H_0. The function $\phi\colon G \to \mathrm{Aut}(\Delta)$ defined via $\phi(g)\colon a \mapsto^{\sigma(g)} a$ has finite image and its quotient $\psi\colon G \to \mathrm{Out}(\Delta)$ is a homomorphism with finite image. Moreover, the pair (d, ϕ) satisfies*

$$^{\phi(g)}\mathrm{d}(h,i)\mathrm{d}(g,hi) = \mathrm{d}(g,h)\mathrm{d}(gh,i)$$

for all $g, h, i \in G$.

Proof. For any $g, h, i \in G$ we calculate

$$\begin{aligned}\mathrm{d}(g,h)\mathrm{d}(gh,i) &= \sigma(g)\sigma(h)\sigma(i)\sigma(ghi)^{-1} = \sigma(g)\mathrm{d}(h,i)\sigma(g)^{-1}\mathrm{d}(g,hi) \\ &=^{\phi(g)} \mathrm{d}(h,i)\mathrm{d}(g,hi)\end{aligned}$$

so (d, ϕ) satisfies the identity of the proposition. Rearranging terms we see that

$$^{\sigma(g)}\mathrm{d}(h,i) = \mathrm{d}(g,h)\mathrm{d}(gh,i)\mathrm{d}(g,hi)^{-1}$$

so $\sigma(g)$ conjugates any $\mathrm{d}(h,i) \in D$ into the finite set $D \cdot D \cdot D^{-1}$. Here, for two sets $A, B \subset H$, we write $A \cdot B = \{a \cdot b \in H \mid a \in A, b \in B\}$ and A^{-1} denotes the set of inverses of A. This shows that Δ is a normal subgroup of H_0, as D generates Δ, and that $\phi\colon G \to \mathrm{Aut}(\Delta)$ has finite image.

To see that the induced map $\psi\colon G \to \mathrm{Out}(\Delta)$ is a homomorphism, let $g, h \in G$ and $a \in \Delta$. Observe that

$$\begin{aligned}^{\phi(g)\phi(h)}a &= \sigma(g)\sigma(h)a\sigma(h)^{-1}\sigma(g)^{-1} \\ &=^{\mathrm{d}(g,h)\ \sigma(gh)}a\end{aligned}$$

and hence $\phi(g) \circ \phi(h)$ and $\phi(gh)$ differ by an inner automorphism. We conclude that $\psi(g) \circ \psi(h) = \psi(gh)$ as elements in $\mathrm{Out}(\Delta)$. So $\psi\colon G \to \mathrm{Out}(\Delta)$ is a homomorphism. This shows Proposition 3.1.3. □

Remark 3.1.4. In light of Proposition 3.1.3 the extra assumption in Theorem A that the conjugation by the quasihomomorphism induces a finite image in $\mathrm{Aut}(N)$ is natural: Given a short exact sequence $1 \to N \to E \to G \to 1$ that admits a quasihomomorphic section $\sigma : G \to E$ one may see that $1 \to \Delta \to E_0 \to G \to 1$ is a short exact sequence where $\Delta = \Delta(\sigma) < N$ and $E_0 = \langle \sigma(G) \rangle < E$ and the map to $\mathrm{Aut}(\Delta)$ has finite image. In fact this assumption is necessary as Example 3.4.2 shows.

Proposition 3.1.5. *Let $\sigma\colon G \to H$ be a quasihomomorphism. Then the map $\tilde{\sigma}\colon G \to H$ defined via*

$$\tilde{\sigma}(g) = \begin{cases} 1 & \text{if } g = 1 \\ \sigma(g) & \text{else} \end{cases}$$

is also a quasihomomorphism.

Proof. An immediate calculation shows that $D(\tilde{\sigma}) \subset D(\sigma) \cup \{1\}$. □

We will use the last proposition to assume that quasihomomorphic sections of extensions satisfy $\sigma(1) = 1$.

We can now show that both definitions of quasihomomorphisms agree.

Proposition 3.1.6. *Let G, H be groups and let $\sigma\colon G \to H$ be a set-theoretic map. Then σ is a quasihomomorphism in the sense of Definition 2.2.8 if and only if it is a quasihomomorphism in the sense of Fujiwara–Kapovich ([FK16]) i.e. if and only if $\bar{D}(\sigma)$ is finite.*

Proof. Recall that for a set-theoretic map $\sigma\colon G \to H$ we defined $\bar{D}(\sigma) \subset H$ as

$$\bar{D}(\sigma) := \{\sigma(h)^{-1}\sigma(g)^{-1}\sigma(gh) \mid g, h \in G\}.$$

Suppose that $\sigma\colon G \to H$ is a quasihomomorphism in the sense of Definition 2.2.8. We start by noting the following easy property.

Claim 3.1.7. *Let $\sigma\colon G \to H$ be a quasihomomorphism with defect group Δ and let $A \subset \Delta$ be a finite subset of Δ. Then the set*

$$\{{}^{\sigma(g)}A \mid g \in G\}$$

is also a finite subset of Δ.

Proof. By Proposition 3.1.3, the set of automorphisms $\{a \mapsto^{\sigma(g)} a \mid g \in G\} \subset \mathrm{Aut}(\Delta)$ is finite. Hence we see that the set $\{^{\sigma(g)}A \mid g \in G\}$ is the image of a finite set of Δ under finitely many automorphisms of Δ and hence a finite subset of Δ. $\square$

Recall that $D = D(\sigma)$, the defect of σ, is defined as $D(\sigma) := \{\sigma(g)\sigma(h)\sigma(gh)^{-1} \mid g, h \in G\}$. Observe that $\mathrm{d}(1, 1) = \sigma(1)\sigma(1)\sigma(1)^{-1} = \sigma(1)$ and hence $\sigma(1) \in D$. Moreover, we see that $\mathrm{d}(g, g^{-1}) = \sigma(g)\sigma(g^{-1})\sigma(1)^{-1}$, hence $\sigma(g)^{-1} \in \sigma(g^{-1}) \cdot D_0$, where $D_0 = \sigma(1)^{-1} \cdot D^{-1} \subset \Delta$, a finite set. Combining the above expressions we see that for every $g, h \in G$,

$$\sigma(h)^{-1}\sigma(g)^{-1}\sigma(gh) \in \sigma(h^{-1})D_0\sigma(g^{-1})D_0D_0^{-1}\sigma((gh)^{-1})^{-1}.$$

Now observe that the set

$$D_1 = \{\sigma(g^{-1})D_0D_0^{-1}\sigma(g^{-1})^{-1} \mid g \in G\} \subset \Delta$$

is finite by Claim 3.1.7. Hence

$$\sigma(h)^{-1}\sigma(g)^{-1}\sigma(gh) \in \sigma(h^{-1})D_0D_1\sigma(g^{-1})\sigma((gh)^{-1}).$$

Using the claim again we see that

$$D_2 = \{\sigma(h^{-1})D_0D_1\sigma(h^{-1})^{-1} \mid h \in G\}$$

is finite and hence that

$$\bar{D}(\sigma) = \{\sigma(h)^{-1}\sigma(g)^{-1}\sigma(gh) \mid g, h \in G\} \subset D_2\sigma(h^{-1})\sigma(g^{-1})\sigma((gh)^{-1}) \subset D_2D$$

so $\bar{D}(\sigma)$ is indeed a finite set. This shows that any quasihomomorphism in the sense of Definition 2.2.8 is a quasihomomorphism in the sense of [FK16].

Now assume that $\sigma\colon G \to H$ is a map such that the set $\bar{D} = \bar{D}(\sigma)$ is finite and let $\bar{\Delta}$ be the group generated by $\bar{D}$.

Just as before we have the following claim:

Claim 3.1.8. *Let $f\colon G \to H$ be a map such that $\bar{D} = \bar{D}(f)$ is finite and let $\bar{\Delta}$ be the group generated by $\bar{D}$. If $A \subset \bar{\Delta}$ if a finite subset of $\bar{\Delta}$ then the set*

$$\{^{\sigma(g)^{-1}}A \mid g \in G\}$$

is also a finite subset of $\bar{\Delta}$.

Proof. This follows from the same argument as for Claim 3.1.7 using Lemma 2.5 of [FK16] instead of Proposition 3.1.3. □

Observe again that $\sigma(1)^{-1} = \sigma(1)^{-1}\sigma(1)^{-1}\sigma(1) \in \bar{D}(\sigma)$ and using that for all $g \in G$, $\sigma(g)^{-1}\sigma(g^{-1})^{-1}\sigma(1) \in \bar{D}$ we see that $\sigma(g) \in \sigma(g^{-1})^{-1}\bar{D}_0$ where $\bar{D}_0 = \sigma(1)\bar{D}$.

Hence for every $g, h \in G$,

$$\sigma(g)\sigma(h)\sigma(gh)^{-1} \in \sigma(g^{-1})^{-1}\bar{D}_0\sigma(h^{-1})^{-1}\bar{D}_0\bar{D}_0^{-1}\sigma(h^{-1}g^{-1})$$

By Claim 3.1.8, we see that the set

$$\bar{D}_1 = \{\sigma(h^{-1})^{-1}\bar{D}_0\bar{D}_0^{-1}\sigma(h^{-1}) \mid h \in G\}$$

is finite and hence

$$\mathrm{d}(g,h) = \sigma(g)\sigma(h)\sigma(gh)^{-1} \in \sigma(g^{-1})^{-1}\bar{D}_0\bar{D}_1\sigma(h^{-1})^{-1}\sigma(h^{-1}g^{-1}).$$

Using the claim once more we see that the set

$$\bar{D}_2 = \{f(g^{-1})^{-1}\bar{D}_0\bar{D}_1 f(g^{-1}) \mid g \in G\}$$

is finite. Finally,

$$\mathrm{d}(g,h) = \sigma(g)\sigma(h)\sigma(gh)^{-1} \in \bar{D}_2\sigma(g^{-1})^{-1}\sigma(h^{-1})^{-1}\sigma(h^{-1}g^{-1}) \subset \bar{D}_2\bar{D}$$

which is a finite set. Hence $D(\sigma)$ is finite. So every quasihomomorphism in the sense of [FK16] is also a quasihomomorphism in the sense of Definition 2.2.8. □

We use Definition 2.2.8 as it is more natural in the context of group extensions.

3.2 Extensions and proof of Theorem A

Recall from the introduction that an extension of a group G by a group N is a short exact sequence

$$1 \to N \to E \to G \to 1$$

and that each such extension induces a homomorphism $\psi\colon G \to \mathrm{Out}(N)$. We will recall the construction of such ψ in Subsection 3.2.1.

In Subsection 3.2.2 we will define *non-abelian cocycles* (see Definition 3.2.2) for group extensions of G by N which induce ψ. Those are certain pairs of functions (e, ϕ) where $\mathrm{e}\colon G \times G \to N$ and $\phi\colon G \to \mathrm{Aut}(N)$.

We will see that every group extension of G by N inducing ψ gives rise to a non-abelian cocycle (e, ϕ) in Proposition 3.2.3. On the other hand every non-abelian cocycle (e, ϕ) gives rise to an extension $1 \to N \to \mathrm{E}(\mathrm{e}, \phi) \to G \to 1$; see Proposition 3.2.4. We will use this correspondence to prove Theorem A in Subsection 3.2.4. The proof will follow the outline of [Bro82], Chapter VI, 6.

3.2.1 Group extensions

Let $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ be an extension of G by N and let $\sigma\colon G \to E$ be any set-theoretic section of $\pi\colon E \to G$. Then $\sigma\colon G \to E$ induces a map $\phi_\sigma\colon G \to \mathrm{Aut}(N)$ via $\phi_\sigma(g)\colon n \mapsto \iota^{-1}({}^{\sigma(g)}\iota(n))$. See Remark 3.1.2 for notation. Let $\sigma'\colon G \to E$ be another section of π. For every $g \in G$, $\pi \circ \sigma(g) = \pi \circ \sigma'(g)$ hence there is an element $\nu(g) \in N$ such that $\sigma'(g) = \nu(g)\sigma(g)$. Let $\phi_{\sigma'}\colon G \to \mathrm{Aut}(N)$ be the induced map to $\mathrm{Aut}(N)$. We see that for every $n \in N$,

$$^{\phi_{\sigma'}(g)}n = ^{\nu(g)} \left(^{\phi_\sigma(g)}n \right)$$

so $\phi_{\sigma'}(g)$ and $\phi_\sigma(g)$ only differ by an inner automorphism. We conclude that the projection $\psi\colon G \to \mathrm{Out}(N)$ of both ϕ_σ and $\phi_{\sigma'}$ is the same map $\psi\colon G \to \mathrm{Out}(N)$. Hence ψ does not depend on the section.

To see that ψ is a homomorphism, let $g, h \in G$. As $\pi(\sigma(g)\sigma(h)\sigma(gh)^{-1}) = 1$, there is an element $\nu(g,h) \in N$ such that $\iota(\nu(g,h)) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$. In particular, for every $n \in N$,

$$^{\phi_\sigma(g)\circ\phi_\sigma(h)}n = ^{\nu(g,h)} \left(^{\phi_\sigma(gh)}n \right)$$

and hence $\phi_\sigma(g) \circ \phi_\sigma(h)$ and $\phi_\sigma(gh)$ only differ by an inner automorphism, so $\psi(g) \circ \psi(h) = \psi(gh)$ and $\psi\colon G \to \mathrm{Out}(N)$ is indeed a homomorphism.

If $1 \to N \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} G \to 1$ and $1 \to N \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} G \to 1$ are two equivalent group extensions (see Definition 3.0.1) with isomorphism $\Phi\colon E_1 \to E_2$ and if $\sigma_1\colon G \to E_1$ is a section of $\pi_1\colon G \to E_1$ then it is easy to see that $\sigma_2 = \Phi \circ \sigma_1\colon G \to E_2$ is a section of $\pi_2\colon E_2 \to G$ and that $\phi_{\sigma_1} = \phi_{\sigma_2}$. Hence the induced homomorphism $\psi\colon G \to \mathrm{Out}(N)$ is the same. We collect these facts in a proposition:

Proposition 3.2.1. *Let* $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ *be a group extension of* G *by* N*. Any two sections* $\sigma, \sigma' \colon G \to E$ *of* π *induce the same homomorphism* $\psi \colon G \to \mathrm{Out}(N)$*. Moreover, two equvivalent group extensions (see Definition 3.0.1) induce the same homomorphism* $\psi \colon G \to \mathrm{Out}(N)$*.*

3.2.2 Non-abelian cocyclces

To show Theorem A we will transform the problem of finding all group extensions of G by N which induce ψ to the problem of finding certain pairs (e, ϕ) called *non-abelian cocycles* where $e \colon G \times G \to N$ and $\phi \colon G \to \mathrm{Aut}(N)$ are certain set-theoretic functions.

Definition 3.2.2. Let G, N be groups and let $\psi \colon G \to \mathrm{Out}(N)$ be a homomorphism. Let $\mathrm{e} : G \times G \to N$ and $\phi \colon G \to \mathrm{Aut}(N)$ be set-theoretic functions such that

(i) $\phi \colon G \to \mathrm{Aut}(N)$ projects to $\psi \colon G \to \mathrm{Out}(N)$, $\phi(1) = 1$ and for all $g \in G$, $\mathrm{e}(1, g) = \mathrm{e}(g, 1) = 1$,

(ii) for all $g, h \in G$ and $n \in N$, ${}^{\mathrm{e}(g,h)}n = {}^{\phi(g)\phi(h)\phi(gh)^{-1}}\, n$ and

(iii) for all $g, h, i \in G$, ${}^{\phi(g)}\mathrm{e}(h, i)\mathrm{e}(g, hi) = \mathrm{e}(g, h)\mathrm{e}(gh, i)$.

Then we say that (e, ϕ) is a *non-abelian cocycle with respect to* (G, N, ψ).

The idea of studying extensions using these non-abelian cocycles is classical; see Chapter IV, 5.6 of [Bro82]. Here, the author simply calls this a "cocycle condition". In order not to confuse it with the cocycle condition of an ordinary 2-cycle we call it "non-abelian cocycle" with respect to the data for group extensions. Consider Remark 3.1.2 for the notation of conjugation and action of automorphisms.

Every group extension $1 \to N \xrightarrow{\iota} E \xrightarrow{\pi} G \to 1$ that induces $\psi \colon G \to \mathrm{Out}(N)$ yields a non-abelian cocycle with respect to (G, N, ψ): As in Subsection 3.2.1, pick a set-theoretic section $\sigma \colon G \to E$ such that $\sigma(1) = 1$, define $\phi_\sigma \colon G \to \mathrm{Aut}(N)$ via ${}^{\phi_\sigma(g)}n = \iota^{-1}\left({}^{\sigma(g)}\iota(n)\right)$ and define $\mathrm{e}_\sigma \colon G \times G \to N$ via $\mathrm{e}_\sigma \colon (g, h) \mapsto \iota^{-1}(\sigma(g)\sigma(h)\sigma(gh)^{-1})$. Observe that σ is a quasihomomorphism if and only if e_σ has finite image.

Proposition 3.2.3. *Let $1 \to N \to E \to G \to 1$ be an extension which induces ψ.*

1. *For any section $\sigma \colon G \to E$ with $\sigma(1) = 1$ the pair $(\mathrm{e}_\sigma, \phi_\sigma)$ is indeed a non-abelian cocycle with respect to (G, N, ψ).*

2. *Let $\phi \colon G \to \mathrm{Aut}(N)$ be a lift of ψ with $\phi(1) = 1$. Then there is a section $\sigma \colon G \to E$ with $\sigma(1) = 1$ such that $\phi_\sigma = \phi$, for ϕ_σ as above. If the extension is in addition bounded (see Definition 3.0.4) and ϕ has finite image, then σ may be chosen to be a quasihomomorphism with $\sigma(1) = 1$.*

Proof. Part (1) is classical and may be found in the proof of Theorem 5.4 of [Bro82].

To see (2), let $\tau \colon G \to E$ be any section of $\pi \colon E \to G$ with $\tau(1) = 1$. Both ϕ and ϕ_τ are lifts of ψ and hence differ only by an inner automorphism. Let $\nu \colon G \to N$ be a representative of such an inner automorphism with $\nu(1) = 1$. Then for every $n \in N$, $g \in G$,

$$ {}^{\phi(g)}n = {}^{\nu(g)} \left({}^{\phi_\tau(g)}n \right) = {}^{(\nu(g)\tau(g))}\, n. $$

Let $\sigma \colon G \to E$ be the section defined via $\sigma(g) = \nu(g)\tau(g)$. Then we see that $\phi = \phi_\sigma$. Assume now that the extension is in addition bounded and that ϕ has finite image. Since the extension is bounded, there is a section $\tau \colon G \to E$ which is a quasihomomorphism and such that $\phi_\tau \colon G \to \mathrm{Aut}(N)$ has finite image. By Proposition 3.1.5 we may assume that $\tau(1) = 1$. We see that we may choose $\nu \colon G \to N$ to also have finite image.

We claim that the section $\sigma \colon G \to E$ defined via $\sigma \colon g \mapsto \nu(g)\tau(g)$ is a quasihomomorphism. Indeed for any $g, h \in G$ we calculate

$$ \begin{aligned} \sigma(g)\sigma(h)\sigma(gh)^{-1} &= \nu(g)\tau(g)\nu(h)\tau(h)\tau(gh)^{-1}\nu(gh)^{-1} \\ &= \nu(g)^{\tau(g)}\nu(h) \left(\tau(g)\tau(h)\tau(gh)^{-1} \right) \nu(gh)^{-1} \\ &\in \mathcal{N}\mathcal{M}D(\tau)\mathcal{N}^{-1} \end{aligned} $$

where $\mathcal{N} = \{\nu(g) \mid g \in G\}$, the image of ν, $\mathcal{M} = \{{}^{\tau(g)}\nu(h) \mid g, h\}$ which is finite. So all sets on the right hand side are finite and hence σ is a quasihomomorphism. This concludes the proof of Proposition 3.2.3. $\square$

3.2.3 Non-abelian cocycles yield group extensions

Let (e, ϕ) be a non-abelian cocycle with respect to (G, N, ψ). We now describe how (e, ϕ) gives rise to a group extension $1 \to N \to \mathrm{E}(\mathrm{e}, \phi) \to G \to 1$ which induces ψ. For this we define a group structure on the set $N \times G$ via

$$(n_1, g_1) \cdot (n_2, g_2) = (n_1 \, {}^{\phi(g_1)}n_2 \mathrm{e}(g_1, g_2), g_1 g_2)$$

for two elements $(n_1, g_1), (n_2, g_2) \in N \times G$. We denote this group by $\mathrm{E}(\mathrm{e}, \phi)$ and define the maps $\iota \colon N \to \mathrm{E}(\mathrm{e}, \phi)$ via $\iota \colon n \mapsto (n, 1)$, $\pi \colon \mathrm{E}(\mathrm{e}, \phi) \to G$ via $\pi \colon (n, g) \mapsto g$ and $\sigma \colon G \to \mathrm{E}(\mathrm{e}, \phi)$ via $\sigma \colon g \mapsto (1, g)$.

Proposition 3.2.4. *Let* (e, ϕ) *be a non-abelian cocycle with respect to* (G, N, ψ) *and let* $\mathrm{E}(\mathrm{e}, \phi)$, $\iota \colon N \to \mathrm{E}(\mathrm{e}, \phi)$, $\pi \colon \mathrm{E}(\mathrm{e}, \phi) \to G$ *and* $\sigma \colon G \to \mathrm{E}(\mathrm{e}, \phi)$ *be as above. Then*

1. $1 \to N \xrightarrow{\iota} \mathrm{E}(\mathrm{e}, \phi) \xrightarrow{\pi} G \to 1$ *is an extension of* G *by* N *inducing* $\psi \colon G \to \mathrm{Out}(N)$. *Moreover,* σ *is a section of* π *such that* $\mathrm{e} = \mathrm{e}_\sigma$ *and* $\phi = \phi_\sigma$.

2. *If both* $\phi \colon G \to \mathrm{Aut}(N)$ *and* $\mathrm{e} \colon G \times G \to N$ *have finite image then the extension we obtain is bounded (see Definition 3.0.4).*

Proof. Part (1) is classical; see Chapter IV.6 of [Bro82] where such extensions from non-abelian cocycles are implicitly constructed.

For part (2), suppose that both e and ϕ have finite image then the section $\sigma \colon G \to \mathrm{E}(\mathrm{e}, \phi)$ is a quasihomomorphism as the defect is just the image of e and, moreover, the map $\phi_\sigma = \phi$ has finite image. Hence the extension is bounded. This concludes the proof of Proposition 3.2.4. □

For the proof of Theorem A we will need to determine when two non-abelian cocycles correspond up to equivalence to the same group extension. We will need the following statement which is stated, though not proved, at the end of IV.6 in [Bro82].

Proposition 3.2.5. *Let* G, N *be groups, let* $\psi \colon G \to \mathrm{Out}(N)$ *be a homomorphism and let* $\phi \colon G \to \mathrm{Aut}(N)$ *be a lift with* $\phi(1) = 1$. *Let* $\mathrm{e}, \mathrm{e}' \colon G \times G \to N$ *be two set-theoretic functions such that for all* $g \in G$, $\mathrm{e}(1, g) = \mathrm{e}(g, 1) = 1$ *and* $\mathrm{e}'(1, g) = \mathrm{e}'(g, 1) = 1$.

1. *If* (e,ϕ) *is a non-abelian cocycle with respect to* (G,N,ψ) *then* (e',ϕ) *is a non-abelian cocycle with respect to* (G,N,ψ) *if and only if there is a map* $\mathrm{c}\colon G\times G\to Z(N)=Z$ *satisfying* $\delta^2\mathrm{c}=0$ *such that for all* $g,h\in G$, $\mathrm{e}'(g,h)=\mathrm{c}(g,h)\cdot\mathrm{e}(g,h)$ *and for all* $g\in G$, $\mathrm{c}(1,g)=\mathrm{c}(g,1)=1$.

2. *If both* (e,ϕ) *and* (e',ϕ) *are non-abelian cocycles with respect to* (G,N,ψ) *then the group extensions corresponding to* (e,ϕ) *and* (e',ϕ) *are equivalent if and only if there is a map* $\mathrm{z}\colon G\to Z=Z(N)$ *with* $\mathrm{z}(1)=1$ *such that* $\mathrm{e}(g,h)=(\delta^1\mathrm{z})(g,h)\mathrm{e}'(g,h)$.

Recall that $Z(N)=Z$ *denotes the centre of* N.

Proof. To see (1), note that for every $g,h\in G$, $n\in N$,

$$ {}^{\mathrm{e}(g,h)}n = {}^{\phi(g)\phi(h)\phi(gh)^{-1}}\, n = {}^{\mathrm{e}'(g,h)}\, n $$

by (ii) of Definition 3.2.2. Hence there is an element $\mathrm{c}(g,h)\in Z(N)$ such that $\mathrm{e}'(g,h)=\mathrm{c}(g,h)\mathrm{e}(g,h)$ and for all $g\in G$, $\mathrm{c}(1,g)=\mathrm{c}(g,1)=1$. Moreover, for every $g,h,i\in G$,

$$ \begin{aligned} {}^{\phi(g)}\mathrm{e}'(h,i)\mathrm{e}'(g,hi) &= \mathrm{e}'(g,h)\mathrm{e}'(gh,i) \\ {}^{\phi(g)}\mathrm{c}(h,i)^{\phi(g)}\mathrm{e}(h,i)\mathrm{c}(g,hi)\mathrm{e}(g,hi) &= \mathrm{c}(g,h)\mathrm{e}(g,h)\mathrm{c}(gh,i)\mathrm{e}(gh,i) \\ (\delta^2\mathrm{c}(g,h,i))^{\phi(g)}\mathrm{e}(h,i)\mathrm{e}(g,hi) &= \mathrm{e}(g,h)\mathrm{e}(gh,i) \\ \delta^2\mathrm{c}(g,h,i) &= 1 \end{aligned} $$

and hence for $\delta^2\mathrm{c}=0$ if we restrict to Z. On the other hand the same calculation shows that if (e,ϕ) is a non-abelian cocycle and $\mathrm{c}\colon G\times G\to Z(N)$ satisfies $\delta^2\mathrm{c}=0$ then (e',ϕ) is a non-abelian cocycle with $\mathrm{e}'(g,h)=\mathrm{c}(g,h)\mathrm{e}(g,h)$.

For (2) suppose that there is a $\mathrm{z}\colon G\to Z$ as in the proposition. Define the map $\Phi\colon \mathrm{E}(\mathrm{e},\phi)\to\mathrm{E}(\mathrm{e}',\phi)$ via $\Phi\colon (n,g)\mapsto(n\mathrm{z}(g),g)$. Then for every $(n_1,g_1),(n_2,g_2)\in \mathrm{E}(\mathrm{e},\phi)$,

$$ \begin{aligned} \Phi((n_1,g_1))\cdot\Phi((n_2,g_2)) &= (n_1\mathrm{z}(g_1),g_1)\cdot(n_2\mathrm{z}(g_2),g_2) \\ &= (n_1^{\phi(g_1)}n_2\mathrm{z}(g_1)^{\phi(g_2)}\mathrm{z}(g_2)\mathrm{e}'(g_1,g_2),g_1g_2) \\ &= (n_1^{\phi(g_1)}n_2\mathrm{z}(g_1g_2)\delta^1\mathrm{z}(g_1,g_2)\mathrm{e}'(g_1,g_2),g_1g_2) \\ &= (n_1^{\phi(g_1)}n_2\mathrm{e}(g_1,g_2)\mathrm{z}(g_1g_2),g_1g_2) \\ &= \Phi((n_1,g_1)\cdot(n_2,g_2)) \end{aligned} $$

and hence Φ is a homomorphism. It is easy to see that Φ is an isomorphism and that Φ fits into the diagram of Definition 3.0.1. Hence the extensions corresponding to (e, ϕ) and (e', ϕ) are equivalent.

On the other hand suppose that the extensions $1 \to N \xrightarrow{\iota} \mathrm{E}(\mathrm{e}, \phi) \xrightarrow{\pi} G \to 1$ and $1 \to N \xrightarrow{\iota'} \mathrm{E}(\mathrm{e}', \phi) \xrightarrow{\pi'} G \to 1$ are equivalent with sections σ, σ' as before with Isomorphism $\Phi \colon \mathrm{E}(\mathrm{e}, \phi) \to \mathrm{E}(\mathrm{e}', \phi)$.

Note that for all $g \in G$, $\pi' \circ \Phi\left((1, g)\right) = g$ and hence the second coordinate of $\Phi((1, g)) \in \mathrm{E}(\mathrm{e}, \phi)$ is g. Define $\mathrm{z} \colon G \to N$ via $\Phi((1, g)) = (\mathrm{z}(g), g)$. Observe that ${}^{\sigma(g)}\iota(n) = ({}^{\phi(g)}n, 1)$ and ${}^{\sigma'(g)}\iota(n) = ({}^{\phi(g)}n, 1)$ and hence $\sigma(g)$ and $\sigma'(g)$ only differ by an element in the centre hence $\mathrm{z}(g) \in Z$. Note that for every $g, h \in G$,

$$\begin{aligned} (\mathrm{e}(g,h), 1) &= \sigma(g)\sigma(h)\sigma(gh)^{-1} \\ \Phi\Big((\mathrm{e}(g,h), 1)\Big) &= \Phi\Big(\sigma(g)\Big) \cdot \Phi\Big(\sigma(h)\Big) \cdot \Phi\Big(\sigma(gh)\Big)^{-1} \\ (\mathrm{e}(g,h), 1) &= (\mathrm{z}(g)^{\phi(g)}\mathrm{z}(h)\mathrm{z}(gh)^{-1}\mathrm{e}'(g,h), 1). \end{aligned}$$

Comparing the last line we see that $\mathrm{e}(g,h) = \delta^1\mathrm{z}(g,h)\mathrm{e}'(g,h)$ which concludes the proposition. □

3.2.4 Proof of Theorem A

We can now prove Theorem A using the correspondence of group extensions with non-abelian cocycles.

Theorem A. *Let G and N be groups and suppose that $Z = Z(N)$, the centre of N, is equipped with a norm $\| \cdot \|$ such that $(Z, \| \cdot \|)$ has finite balls. Furthermore, let $\psi \colon G \to \mathrm{Out}(N)$ be a homomorphism with finite image.*

There is a class $\omega_b = \omega_b(G, N, \psi) \in \mathrm{H}^3_b(G, Z)$ such that $\omega_b = 0$ in $\mathrm{H}^3_b(G, Z)$ if and only if $\mathcal{E}_b(G, N, \psi) \neq \emptyset$ and $c^3(\omega_b) = \omega$ is the obstruction of Theorem 3.0.2. If $\mathcal{E}_b(G, N, \psi) \neq \emptyset$, then the bijection between the sets $\mathrm{H}^2(G, Z)$ and $\mathcal{E}(G, N, \psi)$ described in Theorem 3.0.2 restricts to a bijection between $im(c^2) \subset \mathrm{H}^2(G, Z)$ and $\mathcal{E}_b(G, N, \psi) \subset \mathcal{E}(G, N, \psi)$.

Recall that a normed G-module Z is said to have finite balls if for every $K > 0$ the set $\{z \in Z \mid \|z\| \leq K\}$ is finite. We will split the proof into several claims. Claim 3.2.6 associates to a tuple (G, N, ψ) as in the theorem a function $\zeta \colon G \times G \to$

N which we then use to define the obstruction class $\omega_b = [\mathrm{o}_b] \in \mathrm{H}^3_b(G, Z)$ in Equation (3.2). In Claims 3.2.7 and 3.2.8 we see that o_b is indeed a bounded cocycle and that $\omega_b = [\mathrm{o}_b] \in \mathrm{H}^3_b(G, Z)$ is independent of the choices made. Finally in Claim 3.2.9 we see that ω_b indeed encodes if (bounded) extensions for the data (G, N, ψ) exist. In Claim 3.2.10 we construct a bijection Ψ between $\mathrm{H}^2(G, Z)$ (resp. $\mathrm{im}(c^2)$) and (bounded) extensions.

Let G, N, $\psi\colon G \to \mathrm{Out}(N)$ and $Z, \|\cdot\|$ be as in the theorem. Choose a lift $\phi\colon G \to \mathrm{Aut}(N)$ of ψ with finite image such that $\phi(1) = 1$.

Claim 3.2.6. *There is a function $\zeta\colon G \times G \to N$ such that for all $g, h \in G$, $n \in N$,*

$$^{\zeta(g,h)}n = {}^{\phi(g)\phi(h)\phi(gh)^{-1}}n$$

where ζ has finite image in N and for all $g \in G$, $\zeta(g, 1) = \zeta(1, g) = 1$.

Proof of Claim 3.2.6. For $g, h \in G$ we have that $\psi(g)\psi(h)\psi(gh)^{-1} = 1$, since ψ is a homomorphism. Hence for every $g, h \in G$, the map $\phi(g)\phi(h)\phi(gh)^{-1} \in \mathrm{Aut}(N)$ is an inner automorphism.

As ϕ has finite image in $\mathrm{Aut}(N)$, the function $(g, h) \mapsto \phi(g)\phi(h)\phi(gh)^{-1}$ has finite image in $\mathrm{Inn}(N) < \mathrm{Aut}(N)$. We may find a lift $\zeta\colon G \times G \to N$ of this map such that ζ has finite image and such that $\zeta(1, g) = \zeta(g, 1) = 1$. This shows Claim 3.2.6. ∎

We now define the obstruction class. Define $\mathrm{o}_b\colon G \times G \times G \to N$ so that for all $g, h, i \in G$,

$$^{\phi(g)}\zeta(h, i)\zeta(g, hi) = \mathrm{o}_b(g, h, i)\zeta(g, h)\zeta(gh, i) \tag{3.2}$$

and observe that o_b necessarily has finite image as both $\zeta\colon G \times G \to N$ and $\phi\colon G \to \mathrm{Aut}(N)$ have finite image. Also, observe that $\mathrm{o}_b(g, h, i) = 1$ if one of $g, h, i \in G$ is trivial.

Claim 3.2.7. *The function $\mathrm{o}_b\colon G \times G \times G \to N$ maps to $Z = Z(N) < N$ the centre of N. Moreover, o_b is a non-degenerate bounded cocycle, i.e. $\delta^3 \mathrm{o}_b = 0$.*

Proof of Claim 3.2.7. First we show that o_b maps to the centre of N. Observe that for all $g, h, i \in G$ and $n \in N$,

$$\begin{aligned} {}^{\phi(g)\zeta(h,i)\zeta(g,hi)}n &= {}^{\phi(g)\phi(h)\phi(i)\phi(hi)^{-1}\phi(g)^{-1}}\left({}^{\phi(g)\phi(hi)\phi(ghi)^{-1}}n\right) \\ &= {}^{\phi(g)\phi(h)\phi(i)\phi(ghi)^{-1}}\,n \\ &= {}^{\phi(g)\phi(h)\phi(gh)^{-1}}\left({}^{\phi(gh)\phi(i)\phi(ghi)^{-1}}n\right) \\ &= {}^{\zeta(g,h)\zeta(gh,i)}\,n \end{aligned}$$

and hence ${}^{\phi(g)}\zeta(h,i)\zeta(g,hi)$ and $\zeta(g,h)\zeta(gh,i)$ induce the same map by conjugation on N and hence just differ by an element of the centre so $\mathrm{o}_b(g,h,i) \in Z$. Since ζ and ϕ have finite image, so does o_b, i.e. $\mathrm{o}_b \in C_b^3(G,Z)$ and it is easy to see that o_b is non-degenerate.

To see that o_b satisfies $\delta^3 \mathrm{o}_b = 0$ we calculate

$$ {}^{\phi(g)\phi(h)}\zeta(i,k)\,{}^{\phi(g)}\zeta(h,ik)\zeta(g,hik) $$

for $g, h, i, k \in G$ in two different ways. First observe that

$$\begin{aligned} {}^{\phi(g)\phi(h)}\zeta(i,k)\,{}^{\phi(g)}\zeta(h,ik)\zeta(g,hik) &= {}^{\phi(g)\phi(h)}\zeta(i,k)\left({}^{\phi(g)}\zeta(h,ik)\zeta(g,hik)\right) \\ &= {}^{\phi(g)\phi(h)}\zeta(i,k)\,\mathrm{o}_b(g,h,ik)\zeta(g,h)\zeta(gh,ik) \\ &= \zeta(g,h)\,{}^{\phi(gh)}\zeta(i,k)\,\mathrm{o}_b(g,h,ik)\,\mathrm{o}_b(g,h,ik) \\ &= \zeta(g,h)\zeta(gh,i)\zeta(ghi,k)\,\mathrm{o}_b(g,h,ik)\,\mathrm{o}_b(gh,i,k) \end{aligned}$$

then observe that

$$\begin{aligned} {}^{\phi(g)\phi(h)}\zeta(i,k)\,{}^{\phi(g)}\zeta(h,ik)\zeta(g,hik) &= \left({}^{\phi(g)\phi(h)}\zeta(i,k)\,{}^{\phi(g)}\zeta(h,ik)\right)\zeta(g,hik) \\ &= {}^{\phi(g)}\left(\mathrm{o}_b(h,i,k)\zeta(h,i)\zeta(hi,k)\right)\zeta(g,hik) \\ &= {}^{\phi(g)}\zeta(h,i)\zeta(g,hi)\zeta(ghi,k)\,{}^{\phi(g)}\mathrm{o}_b(h,i,k)\,\mathrm{o}_b(g,hi,k) \\ &= \zeta(g,h)\zeta(gh,i)\zeta(ghi,k)\,\mathrm{o}_b(g,h,i)\,{}^{\phi(g)}\mathrm{o}_b(h,i,k)\,\mathrm{o}_b(g,hi,k). \end{aligned}$$

Finally, comparing these two terms yields

$$ \delta^3 \mathrm{o}_b(g,h,i,k) = {}^{\phi(g)}\mathrm{o}_b(h,i,k) - \mathrm{o}_b(gh,i,k) + \mathrm{o}_b(g,hi,k) - \mathrm{o}_b(g,h,ik) + \mathrm{o}_b(g,h,i) = 0. $$

So o_b indeed defines a bounded cocycle. This shows Claim 3.2.7. □

Claim 3.2.8. *The class* $[\mathrm{o}_b] \in \mathrm{H}^3_b(G, Z)$ *is independent of the choices made for* ζ *and* ϕ.

Proof of Claim 3.2.8. Let $\phi, \phi' \colon G \to \mathrm{Aut}(N)$ be two lifts of ψ as above and choose corresponding functions $\zeta, \zeta' \colon G \to N$ representing the defect of ϕ and ϕ' as above. There is a finite function $\nu \colon G \to N$ with finite image such that $\phi(g) = \bar{\nu}(g)\phi'(g)$ where $\bar{\nu}(g)$ is the element in $Inn(N) \subset \mathrm{Aut}(N)$ corresponding to the conjugation by $\nu(g)$. We calculate

$$\phi(g)\phi(h)\phi(gh)^{-1} = \bar{\nu}(g)^{\phi'(g)}\bar{\nu}(h)\left(\phi'(g)\phi'(h)\phi'(gh)^{-1}\right)\bar{\nu}(gh)^{-1}.$$

We see that for every $n \in N$,

$$\begin{aligned} {}^{\zeta(g,h)}n &= {}^{\phi(g)\phi(h)\phi(gh)^{-1}}\ n \\ &= {}^{\bar{\nu}(g)^{\phi'(g)}\bar{\nu}(h)\left(\phi'(g)\phi'(h)\phi'(gh)^{-1}\right)\bar{\nu}(gh)^{-1}}\ n \\ &= {}^{\nu(g)^{\phi'(g)}\nu(h)\zeta'(g,h)\nu(gh)^{-1}}\ n. \end{aligned}$$

So $\zeta(g,h)$ and $\nu(g)^{\phi'(g)}\nu(h)\zeta'(g,h)\nu(gh)^{-1}$ only differ by an element of the centre. Hence define $z(g,h) \in Z$ via

$$\zeta(g,h) = z(g,h)\nu(g)^{\phi'(g)}\nu(h)\zeta'(g,h)\nu(gh)^{-1}$$

and note that $z : G \times G \to Z$ is a function with finite image as all functions involved in its definition have finite image.

It is a calculation to show that o_b, the obstruction defined via the choices ϕ and ζ and o'_b, the obstruction defined via the choices ϕ' and ζ' differ by $\delta^2 z$ and hence define the same class in bounded cohomology. This shows Claim 3.2.8. □

We call this class $[\mathrm{o}_b] \in \mathrm{H}^3_b(G, Z)$ the *obstruction for extensions G by N inducing* ψ and denote it by $\omega_b(G, N, \psi)$ or ω_b. We have seen that ω_b is a well defined class that depends only on G, N and $\psi \colon G \to \mathrm{Out}(N)$. Next we show that it is an obstruction to (bounded) extensions.

Claim 3.2.9. *Let* $\omega_b \in \mathrm{H}^3_b(G, Z)$ *be as above. Then* $\omega_b = 0 \in \mathrm{H}^3_b(G, Z)$ *if and only if* $\mathcal{E}_b(G, N, \psi) \neq \emptyset$. *Moreover,* $c^3(\omega_b)$ *is equal to the classical obstruction.*

Recall that $c^3 \colon \mathrm{H}^3_b(G, Z) \to \mathrm{H}^3(G, Z)$ denotes the comparison map.

Proof of Claim 3.2.9. Suppose that $c^3(\omega_b) = 0 \in \mathrm{H}^3(G, Z)$. Then there is $\beta \in C^2(G, Z)$ possibly with unbounded, i.e. infinite image, such that

$$\mathrm{o}_b(g, h, i) = ^{\phi(g)} \beta(h, i) - \beta(gh, i) + \beta(g, hi) - \beta(g, h) \tag{3.3}$$

for all $g, h, i \in G$. Moreover we may choose β such that for all $g \in G$, $\beta(1, g) = \beta(g, 1) = 0$ by Proposition 3.1.1 since o_b is non-degenerate.

Define $\mathrm{e}\colon G \times G \to N$ via $\mathrm{e}(g, h) = \zeta(g, h)\beta(g, h)^{-1}$. We will show that (e, ϕ) is a non-abelian cocycle with respect to (G, N, ψ). Indeed, ϕ is a lift of ψ which satisfies $\phi(1) = 1$ and for all $g \in G$, $\mathrm{e}(g, 1) = \mathrm{e}(1, g) = 1$. Moreover, observe that for all $g, h \in G$ and $n \in N$,

$$^{\mathrm{e}(g,h)}n = ^{\zeta(g,h)\beta(g,h)^{-1}} n = ^{\zeta(g,h)} n = ^{\phi(g)\phi(h)\phi(gh)^{-1}} n$$

as $\beta(g, h)$ is in the centre of N. Finally, for all $g, h, i \in G$ we calculate

$$\begin{aligned} ^{\phi(g)}\zeta(h, i)\zeta(g, hi) &= \mathrm{o}_b(g, h, i)\zeta(g, h)\zeta(gh, i) \\ ^{\phi(g)}\left(\zeta(h, i)\beta(h, i)^{-1}\right)\zeta(g, hi)\beta(g, hi)^{-1} &= \zeta(g, h)\beta(g, h)^{-1}\zeta(gh, i)\beta(gh, i)^{-1} \\ ^{\phi(g)}\mathrm{e}(h, i)\mathrm{e}(g, hi) &= \mathrm{e}(g, h)\mathrm{e}(gh, i) \end{aligned}$$

and hence indeed (e, ϕ) is a non-abelian cocycle with respect to (G, N, ψ).

By Proposition 3.2.4, (e, ϕ) gives rise to an extension of G by N which induces ψ and hence $\mathcal{E}(G, N, \psi) \neq \emptyset$.

Analogously, suppose that $\omega_b = 0$ in $\mathrm{H}^3_b(G, Z)$. Then we may find $\beta \in C^3_b(G, Z)$ satisfying Equation (3.3), but with bounded i.e. *finite* image. Hence if we set $\mathrm{e}(g, h) = \zeta(g, h)\beta(g, h)^{-1}$, we see that $\mathrm{e}(g, h)$ has finite image as well, as both ζ and e have. By the above argument (e, ϕ) is a non-abelian cocycle and, as both e and ϕ have finite image, (e, ϕ) gives rise to a bounded extension of (N, G, ψ) by (2) of Proposition 3.2.4. Hence $\mathcal{E}_b(G, N, \psi) \neq \emptyset$.

On the other hand, suppose that $\mathcal{E}(G, N, \psi) \neq \emptyset$. This means that there is some extension $1 \to N \to E \to G \to 1$ of G by N which induces ψ. By Propositin 3.2.3, there is a section $\sigma\colon E \to G$ such that $\phi_\sigma = \phi$ and then $(\mathrm{e}_\sigma, \phi)$ is a non-abelian cocycle with respect to (G, N, ψ).

Observe that for all $g, h \in G$, $n \in N$,

$$^{\mathrm{e}_\sigma(g,h)}n = ^{\phi(g)\phi(h)\phi(gh)^{-1}} n = ^{\zeta(g,h)} n$$

and hence there is an $\beta(g,h) \in Z < N$ such that $\mathrm{e}_\sigma(g,h) = \zeta(g,h)\beta(g,h)^{-1}$. As $(\mathrm{e}_\sigma, \phi)$ satisfies *(iii)* of Definition 3.2.2, we see that for all $g, h, i \in G$

$$\begin{aligned} {}^{\phi(g)}(\mathrm{e}_\sigma(h,i))\mathrm{e}_\sigma(g,hi) &= \mathrm{e}_\sigma(g,h)\mathrm{e}_\sigma(gh,i) \\ {}^{\phi(g)}\left(\zeta(h,i)\beta(g,h)^{-1}\right)\zeta(g,hi)\beta(g,hi)^{-1} &= \zeta(g,h)\beta(g,h)^{-1}\zeta(gh,i)\beta(gh,i)^{-1} \\ {}^{\phi(g)}\zeta(h,i)\zeta(g,hi) &= \Big({}^{\phi(g)}\beta(h,i) - \beta(gh,i) \\ &\quad +\beta(g,hi) - \beta(g,h)\Big)\zeta(g,h)\zeta(gh,i) \end{aligned}$$

so

$$\mathrm{o}_b(g,h,i) =^{\phi(g)} \beta(h,i) - \beta(gh,i) + \beta(g,hi) - \beta(g,h) = \delta^2\beta(g,h,i)$$

and hence $c^3(\omega_b) = 0 \in \mathrm{H}^3(G,Z)$.

Now suppose that $\mathcal{E}_b(G,N,\psi) \neq \emptyset$. This means that there is some extension $1 \to N \to E \to G \to 1$ of G by N which induces ψ and which is in addition bounded. Applying (2) of Proposition 3.2.3 once more we see that there is a section $\sigma\colon G \to E$ such that σ is a quasihomomorphism satisfying that $\sigma(1) = 1$ by Proposition 3.1.5 and $\phi_\sigma = \phi$. As σ is a quasihomomorphism, e_σ has finite image.

As e_σ and ζ have finite image the map $\beta \in C^2(G,Z)$ defined via $\mathrm{e}_\sigma(g,h) = \zeta(g,h)\beta(g,h)^{-1}$ also has finite image and hence $\beta \in C^2_b(G,Z)$. The above calculations show that $\mathrm{o}_b = \delta^2\beta$ and hence $\omega_b = 0$ in $\mathrm{H}^3_b(G,Z)$. This finishes the proof of Claim 3.2.9. □

Now suppose that $\mathcal{E}_b(G,N,\psi) \neq \emptyset$. then there is an extension $1 \to N \to E_0 \to G \to 1$ which induces ψ and a section $\sigma_0\colon G \to E_0$ such that $\phi = \phi_{\sigma_0}$ and $\mathrm{e}_0 := \mathrm{e}_{\sigma_0}$ have finite image and (e_0, ϕ) is a non-abelian cocycle with respect to (G,N,ψ).

Claim 3.2.10. *Let* $\Psi\colon \mathrm{H}^2(G,Z) \to \mathcal{E}(G,N,\psi)$ *be the map defined via*

$$\Psi\colon [\alpha] \mapsto (1 \to N \to \mathrm{E}(\alpha \cdot \mathrm{e}_0, \phi) \to G \to 1)\,,$$

where α *is a non-degenerate representative. Then* Ψ *is a bijection which restricts to a bijection between* $\mathrm{im}(c^2) \subset \mathrm{H}^2(G,Z)$ *and* $\mathcal{E}_b(G,N,\psi) \subset \mathcal{E}(G,N,\psi)$.

Here $\alpha \cdot \mathrm{e}_0$ denotes the map $\alpha \cdot \mathrm{e}_0\colon G \times G \to N$ defined via $\alpha \cdot \mathrm{e}_0\colon (g,h) \mapsto \alpha(g,h) \cdot \mathrm{e}_0(g,h)$.

Proof of Claim 3.2.10. We first show that the above map is well defined: Let $\alpha \in C^2(G, Z)$ be a non-degenerate cocycle. Then $\delta^2\alpha = 0$ and hence by Proposition 3.2.5, $(\alpha \cdot \mathrm{e}_0, \phi)$ is a non-abelian cocycle with respect to (G, N, ψ). If $[\alpha'] = [\alpha]$ in $\mathrm{H}^2(G, Z)$ then there is an element $\mathrm{z} \in C^1(G, Z)$ such that $\alpha = \alpha' + \delta^1\mathrm{z}$. Then, according to point (2) of Proposition 3.2.5, the group extensions are equivalent. Hence Ψ is well defined.

Now suppose that $\Psi([\alpha]) = \Psi([\alpha'])$. Then, according to Proposition 3.2.5 (2) we have that there is a $\mathrm{z} \in C^1(G, Z)$ such that $(\delta^1\mathrm{z})\alpha'\mathrm{e}_0 = \alpha\mathrm{e}_0$ and hence $\delta^1\mathrm{z}\alpha' = \alpha$. Hence $[\alpha] = [\alpha']$ in $\mathrm{H}^2(G, Z)$, so Ψ is injective.

Next we show that Ψ is surjective. Let $1 \to N \to E' \to G \to 1$ be any extension of G by N inducing ψ. By Proposition 3.2.3, there is a section $\sigma' \colon G \to E$ such that $\phi_{\sigma'} = \phi$ and such that (e', ϕ) is a non-abelian cocycle with $\mathrm{e}' = \mathrm{e}_{\sigma'}$. Hence both (e', ϕ) and (e_0, ϕ) are non-abelian cocycles with respect to (G, N, ψ) and by Proposition 3.2.5 there is a map $\beta \in C^2(G, Z)$ such that $\mathrm{e}' = \beta \cdot \mathrm{e}_0$ and $\delta^2\beta = 0$. Then β induces a class and hence $\Psi([\beta])$ corresponds to this extension. This shows that Ψ is surjective and hence that Ψ is a bijection. If $1 \to N \to E' \to G \to 1$ is a bounded extension then we may choose a section $\sigma' \colon G \to E'$ such that e' as above has finite image. Moreover, β as above is bounded as both e' and e_0 are. Hence $[\beta] \in \mathrm{im}(c^2)$ and hence $\Phi(\mathrm{im}(c^2)) \supset \mathcal{E}_b(G, N, \psi)$.

Suppose that $[\alpha] \in \mathrm{im}(c^2)$. Then we may assume that $\alpha \in C^2_b(G, Z)$, i.e. that α has finite image and that α is non-degenerate. Hence $\alpha \cdot \mathrm{e}_0$ has finite image and hene the extension corresponding to $(\alpha \cdot \mathrm{e}_0, \phi)$ is bounded by (2) of Proposition 3.2.4. This shows that $\Psi(\mathrm{im}(c^2)) \subset \mathcal{E}_b(G, N, \psi)$. □

This concludes the proof of Theorem A.

3.3 The set of obstructions and examples

Theorem A provides a characterisation of non-trivial classes $\omega_b \in \mathrm{H}^3_b(G, Z)$, called obstructions. One may wonder which such classes $\omega_b \in \mathrm{H}^3_b(G, Z)$ arise in this way. Recall that in the case of general group extensions, every cocycle in $\mathrm{H}^3(G, Z)$ may be realised as such an obstruction:

Theorem 3.0.3. *For any G-module Z and any $\alpha \in \mathrm{H}^3(G, Z)$ there is a group N with $Z = Z(N)$ and a homomorphism $\psi\colon G \to \mathrm{Out}(N)$ extensing the G-action on Z such that $\alpha = \omega(G, N, \psi)$ in $\mathrm{H}^3(G, N, \psi)$.*

For a normed G-module Z with finite balls and a G-action with finite image define the *set of bounded obstructions* $\mathcal{O}_b(G, Z) \subset \mathrm{H}^3_b(G, Z)$ as

$$\mathcal{O}_b(G, Z) = \{\omega_b(G, N, \psi) \in \mathrm{H}^3_b(G, Z) \mid Z = Z(N), {}^{\psi(g)} z = g\cdot z, \psi : G \to \mathrm{Out}(N) \text{ finite}\}.$$

We refer to the introduction for the definition of $\mathcal{F}(G, Z)$ and observe that Theorem B from the introduction may now be restated as follows:

Theorem B. *Let G be a group and Z be a normed G-module with finite balls and a G-action with finite image. Then*

$$\mathcal{O}_b(G, Z) = \mathcal{F}(G, Z)$$

as subsets of $\mathrm{H}^3_b(G, Z)$.

This fully characterises obstructions we obtain in bounded cohomology.

Proof. We have just seen that $\mathcal{O}_b(G, Z) \subset \mathcal{F}(G, Z)$, as we may choose ω_b in the proof of Theorem A so that it factors through $\mathrm{Out}(N)$ via $\psi\colon G \to \mathrm{Out}(N)$ and $\mathrm{Out}(N)$ is a finite group.

To show $\mathcal{F}(G, Z) \subset \mathcal{O}_b(G, Z)$ we need to show that for every finite group M and any class $\alpha \in \mathrm{H}^3(M, Z)$ there is a group N and a homomorphism $\psi\colon M \to \mathrm{Out}(N)$ which induces α as a cocycle. We recall a construction from [Mac67]. Working with non-degenerate cocycles (see Subsection 2.2.1) we may assume that $\alpha(1, g, h) = \alpha(g, 1, h) = \alpha(g, h, 1) = 0$ for all $g, h \in G$.

Define the abstract symbols $\langle g, h\rangle$ for each $1 \neq g, h \in M$ and set $\langle g, 1\rangle = \langle 1, g\rangle = \langle 1, 1\rangle = 1$ for the abstract symbol 1. Let F be the free group on these symbols and set 1 to be the identity element and set $N = Z \times F$. Define the function $\phi : M \to \mathrm{Aut}(N)$ so that for $g \in M$ the action of $\phi(g)$ on F is given by

$${}^{\phi(g)}\langle h, i\rangle = \alpha(g, h, i)\langle g, h\rangle\langle gh, i\rangle\langle g, hi\rangle^{-1}$$

and so that the action of $\phi(g)$ on Z is given by the M-action on Z. A direct calculation yields that for each $g \in M$, the map $\phi(g)\colon N \to N$ indeed defines an isomorphism. Here, we need the assumption $\alpha(1,g,h) = \alpha(g,1,h) = \alpha(g,h,1) = 0$. It can be seen that for all $n \in N$ and $g_1, g_2 \in F$

$$^{\phi(g_1)\phi(g_2)}n = ^{\langle g_1, g_2\rangle\ \phi(g_1 g_2)}n$$

where we have to use the fact the α is a cocycle. Hence, $\phi\colon M \to \mathrm{Aut}(N)$ is well defined and induces a *homomorphism* $\psi\colon M \to \mathrm{Out}(N)$. It is easy to see that ψ induces the M-action on Z. If $M \ncong \mathbb{Z}_2$, the centre of N is Z. In this case, to calculate $\omega_b(M,N,\psi)$ we choose as representatives for $\phi(g)\phi(h)\phi(gh)^{-1}$ simply $\langle g,h\rangle$ and then see by definition that $\omega_b(M,N,\psi)$ is precisely α.

If $M = \mathbb{Z}_2$ then the centre of N is not Z. However, we can enlarge M by setting $\tilde{M} = M \times \mathbb{Z}_2$. We have both a homomorphism $\pi : \tilde{M} \to M$ via $(m,z) \mapsto m$ and a homomorphism $\iota : M \to \tilde{M}$ via $m \mapsto (m,1)$ such that $\pi \circ \iota = id_M$. Let $\tilde{\alpha} \in \mathrm{H}^3(\tilde{M}, Z)$ be the pullback of α via π. Let $\tilde{N}$ be the group constructed as above with this cocycle and let $\tilde{\phi} : \tilde{M} \to \mathrm{Aut}(\tilde{N})$ and $\tilde{\psi} : \tilde{M} \to \mathrm{Out}(\tilde{N})$ be the corresponding functions. The centre of $\tilde{N}$ is Z. Set $\psi\colon M \to \mathrm{Out}(\tilde{N})$ via $\psi = \tilde{\psi} \circ \iota$. Then the obstruction $\omega_b(M, \tilde{N}, \psi)$ can be seen to be α. This shows Theorem B. □

3.4 Examples and Generalisations

We discuss Examples in Subsection 3.4.1 where we show in particular that the requirements in Definition 3.0.4 are necessary. Subsection 3.4.2 discusses possible generalisations of Theorem A.

3.4.1 Examples

The subset $\mathcal{E}_b(G,N,\psi) \subset \mathcal{E}(G,N,\psi)$ is generally neither empty nor all of $\mathcal{E}(G,N,\psi)$. For any hyperbolic group we have $\mathcal{E}_b(G,N,\psi) = \mathcal{E}(G,N,\psi)$ as the comparison map is surjective ([Min02]). We give different examples where the inclusion $\mathcal{E}_b(G,N,\psi) \subset \mathcal{E}(G,N,\psi)$ is strict.

The examples we discuss will use the *Heisenberg group* $\mathcal{H}_3$. This group fits into the central extension

$$1 \to \mathbb{Z} \to \mathcal{H}_3 \to \mathbb{Z}^2 \to 1.$$

Elements of the Heisenberg group will be described by $[c, z]$, where $c \in \mathbb{Z}$ and $z \in \mathbb{Z}^2$. The group multiplication is given by $[c_1, z_1] \cdot [c_2, z_2] = [c_1 + c_2 + \omega(z_1, z_2), z_1 + z_2]$ where $\omega(z_1, z_2) = \det(z_1, z_2)$, the determinant of the 2×2-matrix (z_1, z_2). Observe that $[c, z]^{-1} = [-c, -z]$, and that ${}^{[c_1, z_1]}[c_2, z_2] = [c_2 + 2\omega(z_1, z_2), z_2]$. The inner automorphisms are isomorphic to $\mathbb{Z}^2$ with the identification $\phi \colon \mathbb{Z}^2 \to \mathrm{Inn}(\mathcal{H}_3)$ via ${}^{\phi(g)}[c, z] = [c + 2\omega(g, z), z]$. It is well-known that ω generates $\mathrm{H}^2(\mathbb{Z}^2, \mathbb{Z})$ and that ω can not be represented by a bounded cocycle, i.e. the comparison map $c^2 \colon \mathrm{H}^2_b(\mathbb{Z}^2, \mathbb{Z}) \to \mathrm{H}^2(\mathbb{Z}^2, \mathbb{Z})$ is trivial.

Example 3.4.1. Let $G = \mathbb{Z}^2$, $N = \mathbb{Z}$ and let $\psi \colon G \to \mathrm{Out}(N)$ be the homomorphism with trivial image. The direct product

$$1 \to N \to N \times G \to G \to 1 \tag{3.4}$$

clearly has a quasihomomorphic section that induces a finite map to $\mathrm{Aut}(N)$ and hence $\mathcal{E}_b(G, N, \psi) \neq \emptyset$. Let $Z(N) = \mathbb{Z}$ be equipped with the standard norm. Note that $\mathrm{im}(c^2) = \{0\}$, for $c^2 \colon \mathrm{H}^2_b(\mathbb{Z}^2, \mathbb{Z}) \to \mathrm{H}^2(\mathbb{Z}^2, \mathbb{Z})$ the comparison map. By Theorem A, $\mathcal{E}_b(\mathbb{Z}^2, \mathbb{Z}, \psi)$ consists of exactly one element, which is the direct product described above. Note that the Heisenberg extension

$$1 \to \mathbb{Z} \to \mathcal{H}_3 \to \mathbb{Z}^2 \to 1$$

is not equivalent to (3.4). This can be seen as $\mathcal{H}_3$ is not abelian. Hence this extension is not bounded. So in this case

$$\emptyset \neq \mathcal{E}_b(\mathbb{Z}^2, \mathbb{Z}, id) \subsetneq \mathcal{E}(\mathbb{Z}^2, \mathbb{Z}, id).$$

Example 3.4.2. The assumption that the quasihomomorphism $\sigma \colon G \to E$ has to induce a map $\phi_\sigma \colon G \to \mathrm{Aut}(N)$ with finite image may seem artificial, as the induced homomorphism $\psi \colon G \to \mathrm{Out}(N)$ already has finite image. However it is necessary as the following example shows.

Consider extensions of $G = \mathbb{Z}^2$ by $N = \mathcal{H}_3$ which induce $\psi \colon G \to \mathrm{Out}(N)$ with trivial image. Again, $\mathcal{E}_b(G, N, \psi)$ is not empty as it contains the extension corresponding to the direct product $1 \to \mathcal{H}_3 \to \mathcal{H}_3 \times \mathbb{Z}^2 \to \mathbb{Z}^2 \to 1$. Moreover, $Z(N) = Z(\mathcal{H}_3) = \mathbb{Z}$ and just as in Example 3.4.1 the comparison map $c^2 \colon \mathrm{H}^2_b(\mathbb{Z}^2, \mathbb{Z}) \to \mathrm{H}^2(\mathbb{Z}^2, \mathbb{Z})$ is trivial, i.e. $\mathrm{im}(c^2) = \{0\}$. So up to equivalence there is just one bounded extension, namely the one corresponding to the direct product $\mathcal{H}_3 \times \mathbb{Z}^2$.

Pick an isomorphism $\phi\colon \mathbb{Z}^2 \to \mathrm{Inn}(\mathcal{H}_3)$. We may construct the extension

$$1 \to \mathcal{H}_3 \to \mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2 \to \mathbb{Z}^2 \to 1 \tag{3.5}$$

where $\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$ denotes the semi-direct product. and observe that the action of $\mathbb{Z}^2$ on the centre of $\mathcal{H}_3$ is trivial as the automorphisms are all inner.

Claim 3.4.3. *The extension (3.5) is not equivalent to the extension* $1 \to \mathcal{H}_3 \to \mathcal{H}_3 \times \mathbb{Z}^2 \to \mathbb{Z}^2 \to 1$.

Proof. Indeed, we show that $\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$ is not isomorphic to $\mathcal{H}_3 \times \mathbb{Z}^2$. We will show that $\mathrm{Z}(\mathcal{H}_3 \times \mathbb{Z}^2) \cong \mathbb{Z}^3$ and $\mathrm{Z}(\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2) \cong \mathbb{Z}$, where $\mathrm{Z}(G)$ denotes the centre of the group G. First, observe that

$$\mathrm{Z}(\mathcal{H}_3 \times \mathbb{Z}^2) \cong \mathrm{Z}(\mathcal{H}_3) \times \mathrm{Z}(Z^2) \cong \mathbb{Z} \times \mathbb{Z}^2 \cong \mathbb{Z}^3.$$

Now assume that $([c, z], n), ([c', z'], n') \in \mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$ are two elements which commute. Then

$$\begin{aligned}
([c, z], n) \cdot ([c', z'], n') &= ([c', z'], n') \cdot ([c, z], n) \\
([c, z] \cdot^n [c', z'], n + n') &= ([c', z'] \cdot^{n'} [c, z], n + n') \\
([c, z] \cdot [c' + 2\omega(n, z'), z'], n + n') &= ([c', z'] \cdot [c + 2\omega(n', z), z], n + n') \\
([c + c' + 2\omega(n, z') + \omega(z, z'), z + z'], n + n') &= ([c' + c + 2\omega(n', z) + \omega(z', z), z + z'], n + n')
\end{aligned}$$

and hence such elements satisfy

$$\omega(n, z') = \omega(n', z) + \omega(z', z) = \omega(n' + z', z).$$

Hence, if $([c, z], n)$ is in the centre of $\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$, then n and z must be such that the above equation holds for every choice of n' and z', and hence $n = z = 0 \in \mathbb{Z}^2$. We conclude that the centre of $\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$ is $\{([c, 0], 0) \mid c \in \mathbb{Z}\}$ and that

$$\mathrm{Z}(\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2) \cong \mathbb{Z}.$$

Hence $\mathcal{H}_3 \times \mathbb{Z}^2$ and $\mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$ cannot be isomorphic. □

So extension (3.5) is not bounded. On the other hand there are two special sort of sections $\sigma\colon G \to \mathcal{H}_3 \rtimes_\phi \mathbb{Z}^2$:

(i) The section $\sigma_1\colon g \mapsto (1, g)$ to (3.5) is a homomorphism and hence in particular a quasihomomorphism. However, the induced map $\phi_{\sigma_1}\colon G \to \mathrm{Aut}(\mathcal{H}_3)$, has as the image the full *infinite* group of inner automorphisms.

(ii) On the other hand, the section $\sigma_2\colon g \mapsto ([1, -g], g)$ induces a trivial map $\phi_{\sigma_2}\colon G \to \mathrm{Aut}(\mathcal{H}_3)$ as seen in the proof of Claim 3.4.3. Indeed we calculate that for $g, h \in G$,

$$\sigma_2(g)\sigma_2(h)\sigma_2(gh)^{-1} = ([\omega(g, h), 0], 0)$$

and so $D(\sigma_2)$ is unbounded and σ_2 is not a quasihomomorphism.

We conclude that there is a section σ_1 which satisfies (i) of Definition 3.0.4 and another section σ_2 which satisfies (ii) of Definition 3.0.4 but no section which satisfies (i) and (ii) simultaneously.

3.4.2 Generalisations

One interesting aspect of Theorem A is that it characterises certain classes in *third bounded cohomology*, namely the obstructions. Moreover we have seen that the obstructions for *bounded* extensions factor through a finite group. Finite groups are amenable and hence all such classes in third bounded cohomology will vanish when passing to real coefficients.

On the other hand every class in third *ordinary* cohomology may be realised by an obstruction; see Theorem 3.0.3. One may wonder if there is another type of extensions $\tilde{\mathcal{E}} \subset \mathcal{E}(G, N, \psi)$ which is empty if and only if a certain class $\tilde{\omega}$ is non-trivial in $\mathrm{H}^3_b(G, \mathbb{R})$. This would be interesting as non-trivial classes in third bounded cohomology with real coefficients are notoriously difficult to construct.

Recall that our Definition 3.0.4 of *bounded* extensions $1 \to N \to E \to G \to 1$ required the existence of sections $\sigma\colon G \to E$ which satisfied two conditions. Namely (i) that σ is a quasihomomorphism, and (ii) that the induced a map $\phi_\sigma\colon G \to \mathrm{Aut}(N)$ by conjugation has finite image. One may wonder if a modification of conditions (i) and (ii) yield different such obstructions with different coefficients. For modifications of (i) there are some generalisations of the quasi-morphisms by Fujiwara–Kapovich, most notably the one by Hartnick–Schweitzer [HS16]. However, there does not seem to be a natural generalisation of condition (ii), i.e. a generalisation of ϕ_σ having finite image. However, such a generalisation is necessary as else the obstructions factor through a finite group and will yield trivial classes with real coefficients. On the other hand, there has to be some restrictions on the sort of sections σ allowed: Consider the bounded cohomology of a free non-abelian group F. Soma [Som97] showed that $\mathrm{H}^3_b(F, \mathbb{R})$ is infinite dimensional. But every extension $1 \to N \to E \to F \to 1$ will even have a homomorphic section $\sigma \colon F \to E$. Without a condition on ϕ_σ there would be no obstruction for such extensions.

Chapter 4

Cup Product in Bounded Cohomology of the Free Group

The material of this chapter is taken from [Heu17a]. This chapter will exclusively focus on the bounded cohomology of non-abelian free groups F with trivial real coefficients, denoted by $\mathrm{H}^n_b(F, \mathbb{R})$. We note that $\mathrm{H}^n_b(F, \mathbb{R})$ is fully unknown for any $n \geq 4$. Free groups play a distinguished rôle in constructing non-trivial classes on other acylindrically hyperbolic groups. Due to a result by Frigerio, Pozzetti and Sisto, any non-trivial alternating class in $\mathrm{H}^n_b(F, \mathbb{R})$ may be promoted to a non-trivial class in $\mathrm{H}^n_b(G, \mathbb{R})$ where G is an acylindrically hyperbolic group and $n \geq 2$; see Corollary 2 of [FPS15].

All classes in the second bounded cohomology of a non-abelian free group F with trivial real coefficients arise as the coboundary of *quasimorphisms* (see Subsection 2.2.5) i.e. for any $\omega \in \mathrm{H}^2_b(F, \mathbb{R})$ there is a quasimorphism $\phi \colon F \to \mathbb{R}$ such that $[\delta^1 \phi] = \omega$. There are many explicit constructions of quasimorphisms $\phi \colon F \to \mathbb{R}$, most prominently the one defined by Brooks [Bro81] and Rolli [Rol09]; see Subsection 2.2.5. One may hope to construct non-trivial classes in $\mathrm{H}^4_b(F, \mathbb{R})$ by taking the cup product $[\delta^1 \phi] \smile [\delta^1 \psi] \in \mathrm{H}^4_b(F, \mathbb{R})$ between two such quasimorphisms $\phi, \psi \colon F \to \mathbb{R}$. We will show that this approach fails.

Theorem C. *Let $\phi, \psi \colon F \to \mathbb{R}$ be two quasimorphisms on a non-abelian free group F where ϕ and ψ are either Brooks counting quasimorphisms on a non self-overlapping word or quasimorphisms in the sense of Rolli. Then $[\delta^1 \phi] \smile [\delta^1 \psi] \in \mathrm{H}^4_b(F, \mathbb{R})$ is trivial.*

We note that Michelle Bucher and Nicolas Monod have independently proved the vanishing of the cup product between the classes induced by Brooks quasimorphisms with a different technique; see [BM18].

Theorem C will follow from a more general vanishing Theorem. For this, we will first define *decompositions* (see Definition 4.1.1) which are certain maps Δ that assign to each element $g \in F$ a finite sequence $(g_1, \dots, g_n)$ of arbitrary length with $g_j \in F$ and such that $g = g_1 \cdots g_n$ and there is no cancellation between the g_j. We then define two new classes of quasimorphisms, namely Δ-*decomposable quasimorphisms* (Definition 4.1.5) and Δ-*continuous quasimorphisms* (Definition 4.1.11). Each Brooks and Rolli quasimorphism will be both Δ-decomposable and Δ-continuous with respect to some decomposition Δ. We will show:

Theorem D. *Let Δ be a decomposition of F and let $\phi, \psi \colon F \to \mathbb{R}$ be quasimorphisms such that ϕ is Δ-decomposable and ψ is Δ-continuous. Then $[\delta^1 \phi] \smile [\delta^1 \psi] \in \mathrm{H}^4_b(F, \mathbb{R})$ is trivial.*

We will prove Theorem D by giving an explicit bounded coboundary in terms of ϕ and ψ in Theorem H. Let ϕ and ψ be as in Theorem D. A key observation of this chapter is that the function $(g, h, i) \mapsto \phi(g)\delta^1\psi(h, i)$ "behaves like a honest cocycle with respect to Δ". The idea of the proof of Theorem H is to mimic the algebraic proof that honest cocycles on free groups have a coboundary; see Subsection 4.2.1.

It was shown by Grigorchuk [Gri95] that Brooks quasimorphisms are *dense* in the vector space of quasimorphisms in the topology of pointwise convergence. In light of Theorem C one would like to deduce from this density that the cup product between all bounded 2-classes vanishes. However, this is not possible. The topology needed for such a deduction is the stronger *defect topology.* Brooks cocycles are not dense in this topology, in fact the space of 2-cocycles is not even separable in this topology. We therefore ask:

Question 4.0.1. Let F be a non-abelian free group. Is the cup product

$$\smile \colon \mathrm{H}^2_b(F, \mathbb{R}) \times \mathrm{H}^2_b(F, \mathbb{R}) \to \mathrm{H}^4_b(F, \mathbb{R})$$

trivial?

Note that it is unknown if nontrivial classes in $\mathrm{H}^4_b(F, \mathbb{R})$ exist. We mention that the cup product on bounded cohomology for other groups need not be trivial. Let G be a group with non-trivial second bounded cohomology. Then $G \times G$ admits a non-trivial cup product

$$\smile : \mathrm{H}^2_b(G \times G, \mathbb{R}) \times \mathrm{H}^2_b(G \times G, \mathbb{R}) \to \mathrm{H}^4_b(G \times G, \mathbb{R})$$

induced by the factors. See [Löh17] for results and constructions in bounded cohomology using the cup product.

Organisation

This chapter is organised as follows: Section 4.1 defines and studies *decompositions* Δ of non-abelian free groups F mentioned above as well as Δ-decomposable and Δ-continuous quasimorphisms. In Section 4.2 we will introduce and prove Theorem H, which will provide the explicit bounded coboundary for the cup products studied in this chapter. The key ideas of the proof are illustrated in Subsection 4.2.1. Theorems C and D will be corollaries of Theorem H and proved in Subsection 4.2.4.

4.1 Decomposition

The aim of this section is to introduce *decompositions of F* in Subsection 4.1.2. Let F be a non-abelian free group with a fixed set of generators. Crudely, a decomposition Δ is a way of assigning a finite sequence $(g_1, \dots g_k)$ of elements $g_j \in F$ to an element $g \in F$ such that $g = g_1 \cdots g_k$ as a reduced word and such that that this decomposition behaves well on geodesic triangles in the Cayley graph. We will see that any decomposition Δ induces a quasimorphism (Proposition 4.1.6), called Δ-decomposable quasimorphism in Subsection 4.1.3. We will introduce special decompositions, Δ_{triv}, Δ_w and Δ_{rolli} and see that Δ_{triv}-decomposable quasimorphisms are exactly the homomorphisms $F \to \mathbb{R}$, that Brooks quasimorphisms on a non self-overlapping word w are Δ_w-decomposable and that the quasimorphisms in the sense of Rolli are Δ_{rolli}-decomposable. In Subsection 4.1.4 we introduce Δ-continuous cocycles.

4.1.1 Notation for sequences

A set $\mathcal{A} \subset F$ will be called *symmetric* if $a \in \mathcal{A}$ implies that $a^{-1} \in \mathcal{A}$. For such a symmetric set $\mathcal{A} \subset F$, we denote by $\mathcal{A}^*$ the set of finite sequences in $\mathcal{A}$ including the empty sequence. This is, the set of all expressions $(a_1, \ldots, a_k)$ where $k \in \mathbb{N}^0$ is arbitrary and $a_j \in \mathcal{A}$. We will denote the element $(a_1, \ldots, a_k) \in \mathcal{A}^*$ by $(\underline{a})$ and k will be called the *length* of $(\underline{a})$ where we set $k = 0$ if $(\underline{a})$ is the empty sequence. For a sequence $(\underline{a})$, we denote by $(\underline{a}^{-1})$ the sequence $(a_k^{-1}, \ldots, a_1^{-1}) \in \mathcal{A}^*$ and the element $\bar{a} \in F$ denotes the product $a_1 \cdots a_k \in F$. We will often work with multi-indexes: The sequences $(\underline{a}_1), (\underline{a}_2), (\underline{a}_3) \in \mathcal{A}^*$ will correspond to the sequences $(\underline{a}_j) = (a_{j,1}, \ldots, a_{j,n_j})$, where n_j is the length of $(\underline{a}_j)$ for $j = 1, 2, 3$. For two sequences $(\underline{a}) = (a_1, \ldots, a_k)$ and $(\underline{b}) = (b_1, \ldots, b_l)$ we define the *common sequence of* $(\underline{a})$ *and* $(\underline{b})$ to be the empty sequence if $a_1 \neq b_1$ and to be the sequence $(\underline{c}) = (a_1, \ldots, a_n)$ where n is the largest integer with $n \leq \min\{k, l\}$ such that $a_j = b_j$ for all $j \leq n$. Moreover, $(\underline{a}) \cdot (\underline{b})$ will denote the sequence $(a_1, \ldots, a_k, b_1, \ldots, b_l)$.

4.1.2 Decompositions of F

We now define the main tool of this chapter, namely *decompositions*. As mentioned in the introduction we will restrict our attention to non-abelian free groups F on a fixed generating set $\mathcal{S}$.

Definition 4.1.1. Let $\mathcal{P} \subset F$ be a symmetric set of elements of F called *pieces* and assume that $\mathcal{P}$ does not contain the identity. A *decomposition of* F *into the pieces* $\mathcal{P}$ is a map $\Delta \colon F \to \mathcal{P}^*$ assigning to every element $g \in F$ a finite sequence $\Delta(g) = (g_1, \ldots, g_k)$ with $g_j \in \mathcal{P}$ such that:

1. For every $g \in F$ and $\Delta(g) = (g_1, \ldots, g_k)$ we have $g = g_1 \cdots g_k$ as a reduced word (no cancelation). Also, we require that $\Delta(g^{-1}) = (g_k^{-1}, \ldots, g_1^{-1})$.

2. For every $g \in F$ with $\Delta(g) = (g_1, \ldots, g_k)$ we have $\Delta(g_i \cdots g_j) = (g_i, \ldots, g_j)$ for $1 \leq i \leq j \leq k$. We refer to this property as Δ being *infix closed*.

3. There is a constant $R > 0$ with the following property.

   Let $g, h \in F$ and let

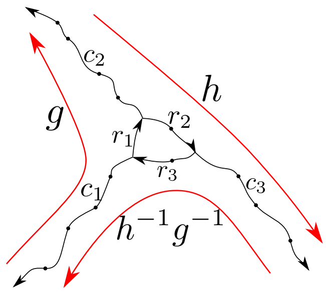


Figure 4.1: $\Delta(g)$, $\Delta(h)$ and $\Delta(h^{-1}g^{-1})$ have sides which can be identified.

- $(\underline{c}_1) \in \mathcal{P}^*$ be such that $(\underline{c}_1^{-1})$ is the common sequence of $\Delta(g)$ and $\Delta(gh)$,
- $(\underline{c}_2) \in \mathcal{P}^*$ be such that $(\underline{c}_2^{-1})$ is the common sequence of $\Delta(g^{-1})$ and $\Delta(h)$ and
- $(\underline{c}_3) \in \mathcal{P}^*$ be such that $(\underline{c}_3^{-1})$ is the common sequence of $\Delta(h^{-1})$ and $\Delta(h^{-1}g^{-1})$.

It is not difficult to see that there are $(\underline{r}_1), (\underline{r}_2), (\underline{r}_3) \in \mathcal{P}^*$ such that

$$\begin{aligned} \Delta(g) &= (\underline{c}_1^{-1}) \cdot (\underline{r}_1) \cdot (\underline{c}_2) \\ \Delta(h) &= (\underline{c}_2^{-1}) \cdot (\underline{r}_2) \cdot (\underline{c}_3) \text{ and} \\ \Delta(h^{-1}g^{-1}) &= (\underline{c}_3^{-1}) \cdot (\underline{r}_3) \cdot (\underline{c}_1). \end{aligned}$$

Then the length of $(\underline{r}_1)$, $(\underline{r}_2)$ and $(\underline{r}_3)$ is bounded by R. See Figure 4.1 for a geometric interpretation and Subsection 4.1.1 for the notation of common sequences and concatenation of sequences.

For such a pair (g, h) we will call $(\underline{c}_1), (\underline{c}_2), (\underline{c}_3)$ the *c-part of the Δ-triangle of* (g, h) and $(\underline{r}_1), (\underline{r}_2), (\underline{r}_3)$ the *r-part of the Δ-triangle of* (g, h). A sequence $(g_1, \ldots, g_k)$ such that

$$\Delta(g_1 \cdots g_k) = (g_1, \ldots, g_k)$$

will be called a *proper Δ sequence*.

Example 4.1.2. Let $\mathcal{S} = \{\mathtt{x}_1, \ldots, \mathtt{x}_n\}$ be an alphabet generating F. Every word $w \in F$ may be uniquely written as a word $w = \mathtt{y}_1 \cdots \mathtt{y}_k$ without backtracking where $\mathtt{y}_i \in \mathcal{S}^{\pm}$. Set $\mathcal{P}_{triv} = \mathcal{S}^{\pm}$ and define the map $\Delta_{triv} \colon F \to \mathcal{P}^*_{triv}$ by setting

$$\Delta_{triv} \colon w \mapsto (\mathtt{y}_1, \ldots, \mathtt{y}_k)$$

for w as above. Then we see that Δ_{triv} is indeed a decomposition. Let $g, h \in G$ and let c_1, c_2, c_3 be such that $g = c_1^{-1}c_2$, $h = c_2^{-1}c_3$ and $gh = c_1^{-1}c_3$ as reduced words. Then the c-part of the Δ_{triv}-triangle of (g,h) is $\Delta_{triv}(c_1), \Delta_{triv}(c_2), \Delta_{triv}(c_3)$ and the r-part of the Δ_{triv}-triangle of (g,h) is $(\emptyset), (\emptyset), (\emptyset)$ where $(\emptyset)$ denotes the empty sequence.

We call the map Δ_{triv} the *trivial decomposition.*

Example 4.1.3. Let $w \in F$ be a non self-overlapping word (see Example 2.2.6). Every word $g \in F$ may be written as $g = u_1 w^{\epsilon_1} u_2 \cdots u_{k-1} w^{\epsilon_{k-1}} u_k$, where the u_j's may be empty, $\epsilon_j \in \{-1,+1\}$ and no u_j contains w or w^{-1} as subwords. It is not hard to show that this expression is unique. Observe that a reduced word in the free group does not overlap with its inverse. Set $\mathcal{P}_w = \{u \in F \mid$ neither w nor w^{-1} are subwords of $u\} \cup \{w, w^{-1}\}$.

We define the *Brooks-decomposition on the word* w as the map $\Delta_w \colon F \to \mathcal{P}_w^*$ by setting

$$\Delta_w \colon g \to (u_1, w^{\epsilon_1}, u_2, \cdots, u_{k-1}, w^{\epsilon_{k-1}}, u_k)$$

for g as above. It is easy to check that this is indeed a decomposition.

Example 4.1.4. As in Example 2.2.7, suppose that F is generated by $\mathcal{S} = \{\mathtt{x}_1, \ldots, \mathtt{x}_n\}$ and observe that every non-trivial element $g \in F$ may be uniquely written as $g = \mathtt{x}_{n_1}^{m_1} \cdots \mathtt{x}_{n_k}^{m_k}$ where all m_j are non-zero and no consecutive n_j are the same. Set $\mathcal{P}_{rolli} = \{\mathtt{x}_j^m \mid j \in \{1, \ldots, n\}, m \in \mathbb{Z}\}$. We define the *Rolli-decompostion* as the map $\Delta_{rolli} \colon F \to \mathcal{P}_{rolli}^*$ via

$$\Delta_{rolli} \colon g \mapsto (\mathtt{x}_{n_1}^{m_1}, \ldots, \mathtt{x}_{n_k}^{m_k})$$

for g as above. It is easy to check that this is indeed a decomposition.

Often we just talk about the decomposition without specifying the pieces $\mathcal{P}$ explicitly. From a decomposition Δ we derive the notion of two sorts of quasimorphisms: Δ-*decomposable quasimorphisms* (Definition 4.1.5) and Δ-*continuous quasimorphisms* (Definition 4.1.11).

4.1.3 Δ-decomposable quasimorphisms

Each decomposition Δ of F induces many different quasimorphisms on F.

Definition 4.1.5. Let Δ be a decomposition with pieces $\mathcal{P}$ and let $\lambda \in \ell^\infty_{alt}(\mathcal{P})$ be a symmetric bounded map on $\mathcal{P}$, i.e. $\lambda(p^{-1}) = -\lambda(p)$ for every $p \in \mathcal{P}$. Then the map $\phi_{\lambda,\Delta} \colon F \to \mathbb{R}$ defined via

$$\phi_{\lambda,\Delta} \colon g \mapsto \sum_{j=1}^{k} \lambda(g_j)$$

where $\Delta(g) = (g_1, \ldots, g_k)$ is called a Δ-*decomposable* quasimorphism.

We may check that such a $\phi_{\lambda,\Delta}$ is indeed a quasimorphism.

Proposition 4.1.6. *Let Δ and λ be as in Definition 4.1.5. Then $\phi_{\lambda,\Delta}$ is a symmetric quasimorphism. If $g, g' \in F$ are such that $\Delta(g \cdot g') = (\Delta(g)) \cdot (\Delta(g'))$ then $\delta^1 \phi(g, g') = 0$. In particular, for all $g \in G$ with $\Delta(g) = (g_1, \ldots, g_k)$ we have that $\delta^1 \phi_{\lambda,\Delta}(g_j, g_{j+1} \cdots g_k) = 0$ for $j = 1, \ldots, k-1$.*

Proof. Symmetry is immediate from the assumptions on $\Delta(g^{-1})$ and λ. Let $g, h \in F$ and let $(\underline{c}_j), (\underline{r}_j)$, $j \in \{1, 2, 3\}$ be as in the definition of the decomposition. We compute

$$\begin{aligned}
\phi_{\lambda,\Delta}(g) &= -\sum_{j=1}^{n_1} \lambda(c_{1,j}) + \sum_{j=1}^{m_1} \lambda(r_{1,j}) + \sum_{j=1}^{n_2} \lambda(c_{2,j}) \\
\phi_{\lambda,\Delta}(h) &= -\sum_{j=1}^{n_2} \lambda(c_{2,j}) + \sum_{j=1}^{m_2} \lambda(r_{2,j}) + \sum_{j=1}^{n_3} \lambda(c_{3,j}) \\
\phi_{\lambda,\Delta}(gh) &= -\sum_{j=1}^{n_1} \lambda(c_{1,j}) - \sum_{j=1}^{m_3} \lambda(r_{3,j}) + \sum_{j=1}^{n_3} \lambda(c_{3,j})
\end{aligned}$$

and hence

$$\delta^1 \phi_{\lambda,\Delta}(g, h) = \phi_{\lambda,\Delta}(g) + \phi_{\lambda,\Delta}(h) - \phi_{\lambda,\Delta}(gh) = \sum_{j=1}^{m_1} \lambda(r_{1,j}) + \sum_{j=1}^{m_2} \lambda(r_{2,j}) + \sum_{j=1}^{m_3} \lambda(r_{3,j})$$

and hence $|\delta^1 \phi_{\lambda,\Delta}(g, h)| \leq 3R \|\lambda\|_\infty$. Note that from this calculation we also see that $\delta^1 \phi_{\lambda,\Delta}(g, h)$ only depends on the r-part of the Δ-triangle for (g, h) and not on the c-part. The second part follows immediately from property (2) of a decomposition. □

Both Brooks and Rolli quasimorphisms are Δ-decomposable quasimorphisms with respect to some Δ as the following examples show:

Example 4.1.7. Let Δ_{triv} be the trivial decomposition of Example 4.1.2. It is easy to see that the Δ_{triv}-decomposable quasimorphisms are exactly the homomorphisms $\phi\colon F \to \mathbb{R}$.

Example 4.1.8. Let $\mathcal{P}_w$ be as in Example 4.1.3 and define $\lambda\colon \mathcal{P}_w \to \mathbb{R}$ by setting

$$\lambda\colon p \mapsto \begin{cases} 1 & \text{if } p = w, \\ -1 & \text{if } p = w^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then it we see that the induced decomposable quasimorphism ϕ_{λ,Δ_w} is the Brooks counting quasimorphism on w; see Example 2.2.6.

Example 4.1.9. Let $\lambda_1, \ldots, \lambda_n$ be as in Example 2.2.7 and let $\mathcal{P}_{rolli}$ be as in Example 4.1.4. Define $\lambda\colon \mathcal{P}_{rolli} \mapsto \mathbb{R}$ by setting

$$\lambda\colon x_j^m \mapsto \lambda_j(m).$$

Then we see that the induced quasimorphism $\phi_{\lambda,\Delta_{rolli}}$ is a Rolli quasimorphism; see Example 2.2.7.

4.1.4 Δ-continuous quasimorphisms and cocycles

We will define Δ-continuous cocycles. Crudely, a cocycle ω is Δ-continuous, if the value $\omega(g,h)$ depends "mostly" on the neighbourhood of the midpoint of the geodesic triangle spanned by e, g, gh in the Cayley graph of F. For this, we will first establish a notion of when two pairs (g,h) and (g',h') of elements in F define triangles which are "close".

For this we define the function $N_\Delta\colon F^2 \times F^2 \to \mathbb{N} \cup \infty$ as follows. Let $(g,h) \in F^2$ and $(g',h') \in F^2$ be two pairs of elements of F. Let $(\underline{c}_j), (\underline{r}_j)$ for $j = 1,2,3$ be the Δ-triangle of (g,h) where $(\underline{c}_j)$ has length n_j and let $(\underline{c}'_j), (\underline{r}'_j)$ for $j = 1,2,3$ be the Δ-triangle of (g',h') where $(\underline{c}'_j)$ has length n'_j.

We set $N_\Delta((g,h),(g',h')) = 0$ if there is a $j \in \{1,2,3\}$ such that $r_j \neq r'_j$ and $N_\Delta((g,h),(g',h')) = \infty$ if $(g,h) = (g',h')$. Else, let $N_\Delta((g,h),(g',h'))$ be the

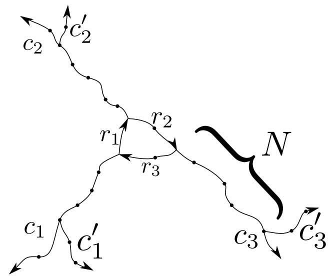


Figure 4.2: The Δ-triangle for (g, h) vs. the Δ-triangle for (g', h') and $N = N_\Delta((g, h), (g', h'))$

largest integer N which satisfies that $N \leq \min\{n_j, n'_j\}$ and $c_{j,k} = c'_{j,k}$ for every $k \leq N$ and $j \in \{1, 2, 3\}$ such that $c_j \neq c'_j$.

Observe that $N_\Delta((g, h), (g', h')) = \infty$ if and only if $(g, h) = (g', h')$. This is because if $(g, h) \neq (g', h')$ then either there is some j such that $r_j \neq r'_j$, in which case $N_\Delta((g, h), (g', h')) = 0$ or there is some j such that $c_j \neq c'_j$ in which case $N_\Delta((g, h), (g', h')) \leq \min\{n_j, n'_j\}$. Crudely, N_Δ measures how much the triangle corresponding to (g, h) agrees with the triangle corresponding to (g', h') arround the "centre" of the triangle; see Figure 4.2. To illustrate N_Δ we will give an example for Δ the trivial decomposition.

Example 4.1.10. Let Δ be the trivial decomposition and let $g, h, i \in F$ be such that ghi has no cancellation and assume that g is not-trivial. Then we claim that $N_\Delta((gh, i), (h, i)) = |h|$, where $|h|$ is the word-length of h. To see this observe that the r-part of the Δ triangles of (gh, i) and (h, i) agrees (it's both $(\emptyset, \emptyset, \emptyset)$. Moreover, the c-part of the Δ-triangles (gh, i) and (h, i) is

$$(\Delta(h)^{-1} \cdot \Delta(g)^{-1}, \emptyset, \Delta(i)) = (\underline{c}_1, \underline{c}_2, \underline{c}_3)$$

and

$$(\Delta(h)^{-1}, \emptyset, \Delta(i)) = (\underline{c}'_1, \underline{c}'_2, \underline{c}'_3).$$

We see that $\underline{c}_2 = \underline{c}'_2$ and $\underline{c}_3 = \underline{c}'_3$ but $\underline{c}_1 \neq \underline{c}'_1$. Observe that the length of $\underline{c}_1$ is $|h| + |g|$ and the length of $\underline{c}'_1$ is $|h|$. Moreover, $c_{1,k} = c'_{1,k}$ for every $k \leq |h|$. This shows that indeed $N_\Delta((gh, i), (h, i)) = |h|$.

Definition 4.1.11. Let Δ be a decomposition of F and let N_Δ be as above. A quasimorphism ϕ is called Δ-continuous if ϕ is symmetric (i.e. $\phi(g^{-1}) = -\phi(g)$ for every $g \in G$) and $\omega = \delta^1\phi$ satisfies that there is a constant $S_{\omega,\Delta} > 0$ and a non-negative summable sequence $(s_j)_{j\in\mathbb{N}}$ with $\sum_{j=0}^{\infty} s_j = S_{\omega,\Delta}$ such that for all $(g,h),(g',h') \in F^2$ we have that either $(g,h) = (g',h')$ or,

$$|\omega(g,h) - \omega(g',h')| \leq s_N$$

where $N = N_\Delta((g,h),(g',h'))$. In this case we call ω Δ-continuous as well.

Crudely, a cocycle ω is Δ-continuous if its values depend mostly on the parts of the decomposition which lies close to the centre of the triangle $g, h, h^{-1}g^{-1}$.

Many quasimorphisms are Δ-continuous as the following proposition shows.

Proposition 4.1.12. *Let Δ be a decomposition of F.*

1. *Every Δ-decomposable quasimorphism is Δ-continuous.*
2. *Every Brooks quasimorphism $\phi\colon F \to \mathbb{R}$ is Δ-continuous.*

Proof. To see (1) observe that the proof of Proposition 4.1.6 shows that $\delta^1\phi(g,h)$ does not depend on the c-part of the Δ-triangle of (g,h). Hence if $N_\Delta((g,h),(g',h')) \geq 1$, then $\delta^1\phi(g,h) = \delta^1\phi(g',h')$.

For (2), suppose that $\delta^1\phi$ is a bounded cocycle induced by a Brooks quasimorphism ϕ on a word w and suppose that the length of w is m. The value of the Brooks cocycle $\delta^1\phi(g,h)$ just depends on the m-neighbourhood of the mid-point of the tripod with endpoints e, g, gh in the Cayley graph. Hence, whenever $N_\Delta((g,h),(g',h')) \geq m$ we have that $\delta^1\phi(g,h) = \delta^1\phi(g',h')$. Note that this implies that Brooks quasimorphisms are Δ-continuous for *any* decomposition Δ. □

4.1.5 Triangles and quadrangles in a tree

Let $g, h \in F$. It is easy to see that there are unique elements $t_1, t_2, d \in F$ such that $g = t_1^{-1}d$ and $h = d^{-1}t_2$ as reduced words and that t_1, t_2 and d are the paths of the tripod with endpoints $1, g, gh$ in the Cayley graph of F. We will call d the *common 2-path* of (g,h).

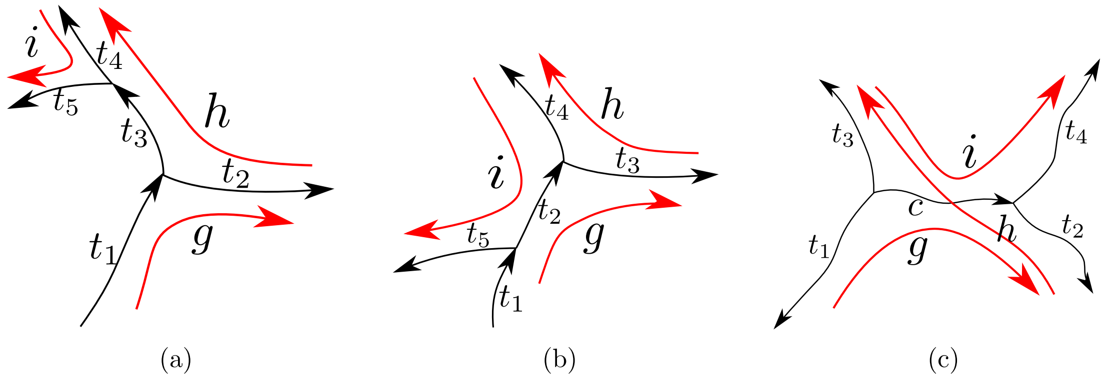


Figure 4.3: Different cases for how g, h and i are aligned

For three elements $g, h, i \in F$ there are three different cases how the geodesics between the points $1, g, gh, ghi$ in the Cayley graph of F can be aligned. See Figure 4.3.

1. (Figure 4.3a): There are elements $t_1, \ldots, t_5$ such that $g = t_1 t_2$, $h = t_2^{-1} t_3 t_4$, $i = t_4^{-1} t_5$ as reduced words.

2. (Figure 4.3b): There are elements $t_1, \ldots, t_5$ such that $g = t_1 t_2 t_3$, $h = t_3^{-1} t_4$, $i = t_4^{-1} t_2^{-1} t_5$ as reduced words.

3. (Figure 4.3c): There are elements $t_1, \ldots, t_4$ and c such that $g = t_1^{-1} c t_2$, $h = t_2^{-1} c^{-1} t_3$, $i = t_3^{-1} c t_4$ as reduced words.

We will say that the *common-3-path* of (g, h, i) is empty in the first two cases and c in the third case.

4.2 Constructing the bounded primitive

Recall that F is a non-abelian free group and let Δ be a decomposition of F; see Definition 4.1.1. Moreover, let $\phi \colon F \to \mathbb{R}$ be a Δ-decomposable quasimorphism (see Definition 4.1.5) and let $\omega \in C_b^2(F, \mathbb{R})$ be a Δ-continuous symmetric 2-cocycle (see Definition 4.1.11). We define the map $\zeta \in C^3(F, \mathbb{R})$ by setting

$$\zeta \colon (g, h, i) \mapsto \sum_{j=1}^{k} \phi(g_j) \omega(g_{j+1} \cdots g_k h, i)$$

for $\Delta(g) = (g_1, \ldots, g_k)$. Moreover, define the maps $\eta, \gamma \in C^2(F, \mathbb{R})$ by setting

- $\eta\colon (g,h) \mapsto \zeta(g, 1, h)$ and
- $\gamma\colon (g,h) \mapsto \frac{1}{2}\Big(\zeta(d, d^{-1}, d) + \zeta(d^{-1}, 1, d)\Big)$ for d the common 2-path of (g,h); see Subsection 4.1.5.

We will show the following theorem:

Theorem H. *Let ϕ be a Δ-decomposable quasimorphism and let ω be a symmetric, Δ-continuous 2-cocycle. Moreover, let γ and η be as above. Then $\beta \in C^3(F, \mathbb{R})$ defined by setting*

$$\beta\colon (g,h,i) \mapsto \phi(g)\omega(h,i) + \delta^2\gamma(g,h,i) + \delta^2\eta(g,h,i)$$

is bounded, i.e. $\beta \in C^3_b(F, \mathbb{R})$.

We will see in Subsection 4.2.4 that β will be the bounded primitive for the cup products studied in this chapter. Before we prove this theorem in Subsection 4.2.3, we will give an idea of the proof in Subsection 4.2.1. This will be inspired by a construction of coboundaries to 3-cocycles which we recall in Subsection 4.2.2.

4.2.1 Idea of the proof of Theorems D and H

Theorem D states that $[\delta^1\phi \smile \omega] = 0$ in $\mathrm{H}^4_b(F, \mathbb{R})$ for ϕ a Δ-decomposable quasimorphism and ω a Δ-continuous cocycle. Equivalently, there is a bounded primitive of the map $\delta^3\tau \in C^3(F, \mathbb{R})$, where τ is given by $\tau\colon (g,h,i) \mapsto \phi(g)\omega(h,i)$ since $\delta^3\tau = \delta^1\phi \smile \omega$. Note that τ is a priori not an interesting function for bounded cohomology: It is neither bounded nor is it a cocycle.

Recall that a map $\alpha \in C^3(F, \mathbb{R})$ satisfies the cocycle condition if and only if for all $g, g', h, i \in F$ we have that

$$\delta^3\alpha(g, g', h, i) = 0.$$

As $\mathrm{H}^3(F, \mathbb{R}) = 0$, we know that there is some $\epsilon \in C^2(F, \mathbb{R})$ such that $\delta^2\epsilon = \alpha$. We will give a purely algebraic construction of such an ϵ in terms of α, provided α satisfies certain weak conditions stated in Subsection 4.2.2, Equation 4.1.

Observe that τ does not satisfy the cocycle condition for *all* $g, g', h, i \in F$. However, τ satisfies the cocycle condition in certain cases: Proposition 4.1.6 shows that if $g, g' \in F$ satisfy that $\Delta(g \cdot g') = (\Delta(g)) \cdot (\Delta(g'))$ then

$$\delta^3\tau(g, g', h, i) = 0$$

for all $h, i \in F$. Following the techniques of Subsection 4.2.2 we will construct an $\epsilon \in C^2(F, \mathbb{R})$ such that $\delta^2\epsilon$ is *boundedly close* to τ. This is, such that the map $\beta = \tau - \delta^2\epsilon$ is bounded, i.e. $\beta \in C_b^3(F, \mathbb{R})$. This will imply that

$$\delta^3\beta = \delta^3\tau - \delta^3\delta^2\epsilon = \delta^1\phi \smile \omega$$

and hence the cup product has a bounded primitive and is trivial in bounded cohomology.

4.2.2 Constructing 2-coboundaries from 3-cocycles

Let $\alpha \in C^3(F, \mathbb{R})$ be a 3-cocycle i.e. a map such that $\delta^3\alpha = 0$. We will show how to construct a map $\epsilon \in C^2(F, \mathbb{R})$ such that $\delta^2\epsilon = \alpha$. We emphasize that this subsection just motivates the strategy of the proof of Theorem H. This theorem will be proved in detail in Subsection 4.2.3 and the proof can be understood without reading this subsection. In both subsections, the η and the ζ term will play analogous rôle.

To simplify our calculations we will assume that α is a cocycle and moreover satisfies

$$\alpha(g, h, 1) = \alpha(g, 1, h) = \alpha(1, g, h) = \alpha(g, g^{-1}, h) = 0 \text{ for all } g, h \in F. \tag{4.1}$$

We note that *alternating cochains* in the sense of Subsection 4.10 of [Fri17] satisfy (4.1) and that such maps may be used to fully compute $\mathrm{H}^3(F, \mathbb{R})$.

Let α be as above and recall that the cocycle condition implies that for all $g, g', h, i \in F$ we have that

$$\delta^3\alpha(g, g', h, i) = \alpha(g', h, i) - \alpha(gg', h, i) + \alpha(g, g'h, i) - \alpha(g, g', hi) + \alpha(g, g', h) = 0. \tag{4.2}$$

In a first step we see how α may be rewritten as a sum of elements of the form $\alpha(\mathtt{x}, g', h')$, where $\mathtt{x}$ is a letter and $g', h' \in F$. Define $\zeta \in C^3(F, \mathbb{R})$ by setting

$$\zeta\colon (g, g', h) \mapsto \sum_{j=1}^{k} \alpha(\mathtt{x}_j, \mathtt{x}_{j+1} \cdots \mathtt{x}_k g', h)$$

where $g = \mathtt{x}_1 \cdots \mathtt{x}_k$ is the reduced word representing g. We claim that

Claim 4.2.1. *Let $\alpha \in C^3(F, \mathbb{R})$ be a cocycle satisfying (4.1). Then*

$$\alpha(g, h, i) = \zeta(g, h, i) - \zeta(g, 1, hi) + \zeta(g, 1, h)$$

for all $g, h, i \in F$.

Proof. direct computation. □

Now define $\eta \in C^2(F, \mathbb{R})$ by setting

$$\eta\colon (g, h) \mapsto \zeta(g, 1, h).$$

We then see that

$$\alpha(g, h, i) + \delta^2\eta(g, h, i) \quad = \quad \zeta(g, h, i) + \zeta(h, 1, i) - \zeta(gh, 1, i)$$

for all $g, h, i \in F$.

Claim 4.2.2. *We have that*

$$\zeta(g, h, i) + \zeta(h, 1, i) - \zeta(gh, 1, i) = \zeta(d, h, i) + \zeta(d^{-1}, dh, i)$$

for all $g, h, i \in F$, where d is the common 2-path of (g, h).

Proof. We will prove this by an explicit calculation. Observe that it is immediate that if $u, v \in F$ are such that uv is reduced then

$$\zeta(uv, g', h) = \zeta(u, vg', h) + \zeta(v, g', h). \tag{4.3}$$

Now rewrite $g = t_1^{-1} d$ and $h = d^{-1} t_2$, where d is the common 2-path of (g, h); see Subsection 4.1.5. Then by (4.3) we see that

- $\zeta(g, h, i) = \zeta(t_1^{-1}, dh, i) + \zeta(d, h, i)$

- $\zeta(h, 1, i) = \zeta(d^{-1}, dh, i) + \zeta(t_2, 1, i)$
- $\zeta(gh, 1, i) = \zeta(t_1^{-1}, dh, i) + \zeta(t_2, 1, i)$

Hence
$$\zeta(g, h, i) + \zeta(h, 1, i) - \zeta(gh, 1, i) = \zeta(d, h, i) + \zeta(d^{-1}, dh, i).$$

□

Claim 4.2.3. *We have that* $\zeta(g, h, i) + \zeta(h, 1, i) - \zeta(gh, 1, i) = 0$ *and hence that* $\alpha(g, h, i) = \delta^2 \epsilon$ *for* $\epsilon = -\eta$.

Proof. Let d be the common 2-path of (g, h) as above. Moreover, suppose that $\mathtt{d}_1 \cdots \mathtt{d}_l$ is the word representing d. By the previous claim, $\zeta(g, h, i) + \zeta(h, 1, i) - \zeta(gh, 1, i) = \zeta(d, h, i) + \zeta(d^{-1}, dh, i)$. We calculate
$$\zeta(d, h, i) + \zeta(d^{-1}, dh, i) = \sum_{j=1}^{k} \Big(\alpha(\mathtt{d}_j, \mathtt{d}_{j+1} \cdots \mathtt{d}_l h, i) + \alpha(\mathtt{d}_j^{-1}, \mathtt{d}_j \cdots \mathtt{d}_l h, i) \Big).$$
By evaluating $\delta^3 \alpha(\mathtt{d}_j, \mathtt{d}_j^{-1}, \mathtt{d}_j \cdots \mathtt{d}_l h, i)$ using property (4.1) we have that
$$\alpha(\mathtt{d}_j^{-1}, \mathtt{d}_j \cdots \mathtt{d}_l h, i) + \alpha(\mathtt{d}_j, \mathtt{d}_{j+1} \cdots \mathtt{d}_l h, i) = 0.$$

□

Together with Claim 4.2.2 the previous claim implies that $\alpha + \delta^2 \eta = \alpha - \delta^2 \epsilon = 0$.

4.2.3 Proof of Theorem H

Let Δ be a decomposition of F (Definition 4.1.1), let ϕ be a Δ-decomposable quasimorphism (Definition 4.1.5) and let ω be a Δ-continuous cocycle (Definition 4.1.11). See the previous subsection for a brief discussion on the classical computations that inspired our construction here. Analogously to Claim 4.2.1, we will first rewrite the function $(g, h, i) \mapsto \phi(g)\omega(h, i)$ as sum of terms $\phi(g_j)\omega(g', h')$ where g_j will be a piece of a fixed decomposition Δ. We will construct a map $\epsilon \in C^2(F, \mathbb{R})$ such that $\delta^2 \epsilon$ is boundedly close to $(g, h, i) \mapsto \phi(g)\omega(h, i)$ by "treating" this function as a cocycle on the pieces of Δ and then performing the calculations of Subsection 4.2.2. For this, define $\zeta \in C^3(F, \mathbb{R})$ by setting
$$\zeta(g, g', h) := \sum_{j=1}^{k} \phi(g_j)\omega(g_{j+1} \cdots g_k g', h)$$

for $\Delta(g) = (g_1, \ldots, g_k)$. Analogous to Claim 4.2.1 we show:

Proposition 4.2.4. *The term $\phi(g)\omega(h,i)$ is equal to*

$$\zeta(g,h,i) - \zeta(g,1,hi) + \zeta(g,1,h)$$

for $\zeta \in C^3(F,\mathbb{R})$ are as above.

Proof. Let $\Delta(g) = (g_1, \ldots, g_k)$. Observe that for all $j \in \{1, \ldots, k\}$ by Proposition 4.1.6 we have that

$$\begin{aligned} 0 &= \delta^1\phi(g_j, g_{j+1}\cdots g_k)\omega(h,i) \\ &= \phi(g_{j+1}\cdots g_k)\omega(h,i) - \phi(g_j g_{j+1}\cdots g_k)\omega(h,i) + \phi(g_j)\omega(g_{j+1}\cdots g_k h, i) + \ldots \\ &\quad -\phi(g_j)\omega(g_{j+1}\cdots g_k, hi) + \phi(g_j)\omega(g_{j+1}\cdots g_k, h) \end{aligned}$$

Rearranging terms we see that

$$\begin{aligned} &\phi(g_j\cdots g_k)\omega(h,i) - \phi(g_{j+1}\cdots g_k)\omega(h,i) \\ = \ &\phi(g_j)\omega(g_{j+1}\cdots g_k h, i) - \phi(g_j)\omega(g_{j+1}\cdots g_k, hi) + \phi(g_j)\omega(g_{j+1}\cdots g_k, h). \end{aligned}$$

Summing for $j = 1, \ldots, k-1$ over both sides

$$\begin{aligned} &\phi(g_1\cdots g_k)\omega(h,i) - \phi(g_k)\omega(h,i) \\ = &\sum_{j=1}^{k-1} \Big(\phi(g_j)\omega(g_{j+1}\cdots g_k h, i) - \phi(g_j)\omega(g_{j+1}\cdots g_k, hi) + \phi(g_j)\omega(g_{j+1}\cdots g_k, h)\Big). \end{aligned}$$

As ω was supposed to be symmetric we have that $\omega(1,h) = \omega(1,hi) = 0$ and hence

$$\phi(g)\omega(h,i) = \zeta(g,h,i) - \zeta(g,1,hi) + \zeta(g,1,h).$$

□

As in Subsection 4.2.2 define $\eta \in C^2(F,\mathbb{R})$ by setting

$$\eta\colon (g,h) \mapsto \zeta(g,1,h)$$

and note that

$$\begin{aligned} \delta^2\eta(g,h,i) &= \eta(h,i) - \eta(gh,i) + \eta(g,hi) - \eta(g,h) \\ &= \zeta(h,1,i) - \zeta(gh,1,i) + \zeta(g,1,hi) - \zeta(g,1,h). \end{aligned}$$

Using Proposition 4.2.4 we see that

$$\phi(g)\omega(h,i) + \delta^2\eta(g,h,i)$$

is equal to

$$\zeta(g,h,i) + \zeta(h,1,i) - \zeta(gh,1,i).$$

We will need the following properties of ζ.

Proposition 4.2.5. *The function ζ defined as above has the following properties.*

1. *If $u_1, u_2, v, w \in F$ are such that u_1u_2 is reduced then $\zeta(u_1u_2, w, v) - \zeta(u_1, u_2w, v) - \zeta(u_2, w, v)$ is uniformly bounded.*
2. *Let $u_1, u_2, u_3, u_4 \in F$ be elements such that u_1u_2, u_2u_3 and u_2u_4 are reduced and u_3 and u_4 do not start with the same letter. Then*

$$\zeta(u_1^{-1}, u_1u_2u_3, u_3^{-1}u_4) + \zeta(u_1, u_2u_3, u_3^{-1}u_4)$$

   is uniformly bounded.
3. *Let $u, v_1, v_2 \in F$ such that v_1uv_2 is reduced. Then*
   (a) $\zeta(u, u^{-1}v_1^{-1}, v_1uv_2) - \zeta(u, u^{-1}, u)$ *and*
   (b) $\zeta(u^{-1}, v_1^{-1}, v_1uv_2) - \zeta(u^{-1}, 1, u)$

   are uniformly bounded.

Proof. In the proof of items (1)-(3) we will frequently use the following claim:

Claim 4.2.6. *Let $u, v_1, v_2 \in F$ be such that v_1uv_2 is reduced, let $\Delta(u) = (u_1, \ldots, u_n)$ and let R be as in Definition 4.1.1. Then, there are sequences $(\underline{v}'_1), (\underline{v}'_2)$ such that*

1. *for every $1 \leq j \leq n - R$ we have that $\Delta(u_j \cdots u_nv_2) = (u_j, \ldots, u_{n-R}) \cdot (\underline{v}'_2)$ and*
2. *for every $R \leq j \leq n$ we have that $\Delta(v_1 \cdot u_1 \cdots u_j) = (\underline{v}'_1) \cdot (u_R, \ldots, u_j)$.*

Proof. For (1) let $(\underline{c}_1), (\underline{c}_2), (\underline{c}_3)$ be the c-part of the Δ-triangle of (u, v_2) and let $(\underline{r}_1), (\underline{r}_2), (\underline{r}_3)$ be the r-part of the Δ-triangle of (u, v_2). Then, as uv_2 is reduced we see that $(\underline{c}_2) = \emptyset$. Hence $\Delta(u) = (\underline{c}_1)^{-1} \cdot (\underline{r}_1)$ and $\Delta(uv) = (\underline{c}_1)^{-1} \cdot (\underline{r}_3) \cdot (\underline{c}_3)$. Moreover, observe that the length of $(\underline{r}_1)$ is bounded by R. Hence all of $(u_1, \ldots, u_{n-R})$ lie in $(\underline{c}_1)^{-1}$. Comparing $\Delta(uv)$ with $\Delta(u)$ and using that decompositions are infix closed (see Definition 4.1.1) yields (1). Item (2) can be deduced by the same argument. □

We first show (1) of Proposition 4.2.5. Let $u_1, u_2 \in F$ be such that u_1u_2 is reduced. Let the c-part of the Δ-triangle of (u_1, u_2) be $(\underline{c}_1), (\underline{c}_2), (\underline{c}_3)$ and let the r-part of the Δ-triangle of (u_1, u_2) be $(\underline{r}_1), (\underline{r}_2), (\underline{r}_3)$. As u_1u_2 is reduced, $(\underline{c}_2)$ has to be empty. Hence

- $\Delta(u_1) = ((\underline{c}_1)^{-1} \cdot (\underline{r}_1))$,
- $\Delta(u_2) = ((\underline{r}_2) \cdot (\underline{c}_3))$ and
- $\Delta(u_1u_2) = ((\underline{c}_1)^{-1} \cdot (\underline{r}_3)^{-1} \cdot (\underline{c}_3))$.

Suppose that $(\underline{c}_i) = (c_{i,1}, \ldots c_{i,n_i})$ and that $(\underline{r}_i) = (r_{i,1}, \ldots r_{i,m_i})$ for $i = 1, 2, 3$. Then

$$\begin{aligned}
\zeta(u_1u_2, w, v) &= \sum_{j=1}^{n_1} \phi(c_{1,j}^{-1})\omega(c_{1,j-1}^{-1} \cdots c_{1,1}^{-1}\bar{r}_3^{-1}\bar{c}_3 w, v) + \sum_{j=1}^{m_3} \phi(r_{3,j}^{-1})\omega(r_{3,j-1}^{-1} \cdots r_{3,1}^{-1}\bar{c}_3 w, v) + \\
&\quad + \sum_{j=1}^{n_3} \phi(c_{3,j})\omega(c_{3,j+1} \cdots c_{3,n_3} w, v) \\
\zeta(u_1, u_2 w, v) &= \sum_{j=1}^{n_1} \phi(c_{1,j}^{-1})\omega(c_{1,j-1}^{-1} \cdots c_{1,1}^{-1}\bar{r}_1 u_2 w, v) + \sum_{j=1}^{m_1} \phi(r_{1,j})\omega(r_{1,j+1} \cdots r_{1,n_1} u_2 w, v) \\
\zeta(u_2, w, v) &= \sum_{j=1}^{m_2} \phi(r_{2,j})\omega(r_{2,j+1} \cdots r_{2,m_2}\bar{c}_3 w, v) + \sum_{j=1}^{n_3} \phi(c_{3,j})\omega(c_{3,j+1} \cdots c_{3,n_3} w, v)
\end{aligned}$$

and hence

$$\begin{aligned}\zeta(u_1u_2,w,v)-\zeta(u_1,u_2w,v)-\zeta(u_2,w,v) &= \sum_{j=1}^{m_3}\phi(r_{3,j}^{-1})\omega(r_{3,j-1}^{-1}\cdots r_{3,1}^{-1}\bar{c}_3w,v)\\ &\quad -\sum_{j=1}^{m_1}\phi(r_{1,j})\omega(r_{1,j+1}\cdots r_{1,n_1}u_2w,v)\\ &\quad -\sum_{j=1}^{m_2}\phi(r_{2,j})\omega(r_{2,j+1}\cdots r_{2,m_2}\bar{c}_3w,v)\end{aligned}$$

which is indeed uniformly bounded, as $m_1, m_2, m_3 \le R$ (see Definition 4.1.1). Since ϕ is Δ-decomposable, ϕ is uniformly bounded on pieces and as ω is a bounded function.

To see (2), let u_1, u_2, u_3, u_4 be as in the proposition and suppose that $\Delta(u_1) = (u_{1,1}, \dots, u_{1,n})$.

Claim 4.2.7. *We have that the r-part of the Δ-triangles of $(u_{1,j}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4)$ are the same for any $j \le n-R$ and that the c-part of $(u_{1,j}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4)$ is $(\underline{c}_1')\cdot(u_{1,n}^{-1},\cdots,u_{1,j}^{-1}),(\underline{c}_2'),(\underline{c}_3')$ for appropriate sequences $(\underline{c}_1'),(\underline{c}_2'),(\underline{c}_3')$. In particular there is a $C \in \mathbb{N}$ such that*

$$N_\Delta((u_{1,j}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4),(u_{1,j+1}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4)) = j + C$$

for all $j \le n-R$.

Proof. It follows by comparing the sequences $\Delta(u_{1,j}\cdots u_{1,n}u_2u_3)$ and $\Delta(u_{1,j}\cdots u_{1,n}u_2u_4)$ using Claim 4.2.6. □

For (2) of Proposition 4.2.5, we calculate

$$\begin{aligned}\zeta(u_1^{-1},u_1u_2u_3,u_3^{-1}u_4)+\zeta(u_1,u_2u_3,u_3^{-1}u_4) &= \sum_{j=1}^{n}\phi(u_{1,j}^{-1})\omega(u_{1,j}\cdots u_{1,n}u_2u_3,u_3^{-1}u_4)\dots\\ &\quad +\sum_{j=1}^{n}\phi(u_{1,j})\omega(u_{1,j+1}\cdots u_{1,n}u_2u_3,u_3^{-1}u_4)\\ &= \sum_{j=1}^{n}\phi(u_{1,j})\Big(\omega(u_{1,j+1}\cdots u_{1,n}u_2u_3,u_3^{-1}u_4)\dots\\ &\quad -\omega(u_{1,j}\cdots u_{1,n}u_2u_3,u_3^{-1}u_4)\Big).\end{aligned}$$

Hence we conclude that $\zeta(u_1^{-1}, u_1u_2u_3, u_3^{-1}u_4) + \zeta(u_1, u_2u_3, u_3^{-1}u_4)$ is uniformly close to

$$\sum_{j=1}^{n-R} \phi(u_{1,j})\Big(\omega(u_{1,j+1}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4) - \omega(u_{1,j}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4)\Big)$$

as R just depends on Δ and ϕ is uniformly bounded on pieces. Now let $(s_j)_{j\in\mathbb{N}}$ be the sequence in Definition 4.1.11. By Claim 4.2.7,

$$|\omega(u_{1,j+1}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4) - \omega(u_{1,j}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4)| < s_{n+C}.$$

and hence

$$\sum_{j=1}^{n-R} |\omega(u_{1,j+1}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4) - \omega(u_{1,j}\cdots u_{1,n}u_2u_3, u_3^{-1}u_4)| < S_{\omega,\Delta}.$$

Putting those estimations together we see that $\zeta(u_1^{-1}, u_1u_2u_3, u_3^{-1}u_4) + \zeta(u_1, u_2u_3, u_3^{-1}u_4)$ is indeed uniformly bounded.

To see (3a), let u, v_1, v_2 be as in the proposition and suppose that $\Delta(u) = (u_1, \dots, u_n)$. By Claim 4.2.6, we see that for $n - R \leq j$ and $R \leq j$ the r-part of the Δ-triangle of $(u_j^{-1}\cdots u_1^{-1}v_1^{-1}, v_1uv_2)$ is trivial and that there are sequences $(\underline{v}_1'), (\underline{v}_2')$ such that the c-part of the Δ-triangle is $\emptyset, (u_j^{-1}, \dots, u_1^{-1}) \cdot (\underline{v}_1'), (u_{j-1}, \dots u_n) \cdot (\underline{v}_2')$. Hence there are integers C_1, C_2 such that

$$N_\Delta\Big((u_j^{-1}\cdots u_1^{-1}v_1^{-1}, v_1uv_2), (u_j^{-1}\cdots u_1^{-1}, u)\Big) \geq \min\{j - R + C_1, n - j - R + C_2\}$$

and hence

$$\sum_{j=R}^{n-R} |\omega(u_j^{-1}\cdots u_1^{-1}v_1^{-1}, v_1uv_2) - (u_j^{-1}\cdots u_1^{-1}, u)| \leq 2S_{\omega,\Delta}.$$

Finally, observe that

$$\zeta(u, u^{-1}v_1^{-1}, v_1uv_2) - \zeta(u, u^{-1}, u) = \sum_{j=1}^{n} \phi(u_j)\Big(\omega(u_j^{-1}\cdots u_1^{-1}v_1^{-1}, v_1uv_2) - (u_j^{-1}\cdots u_1^{-1}, u)\Big)$$

and is hence uniformly close to

$$\sum_{j=R}^{n-R} \phi(u_j)\Big(\omega(u_j^{-1}\cdots u_1^{-1}v_1^{-1}, v_1uv_2) - (u_j^{-1}\cdots u_1^{-1}, u)\Big)$$

With the above estimation we hence see that $\zeta(u, u^{-1}v_1^{-1}, v_1uv_2) - \zeta(u, u^{-1}, u)$ may be uniformly bounded. The proof of item (3b) is analogous to the proof for item (3a). $\square$

Analogously to Claim 4.2.2 we show:

Proposition 4.2.8. *The term $\phi(g)\omega(h,i)+\delta^2\eta(g,h,i)$ is uniformly close to $\zeta(d,h,i)+\zeta(d^{-1},dh,i)$ where d is the common 2-path of (g,h).*

Proof. Let $g, h, i \in F$. Furthermore write $g = t_1^{-1}d$ and $h = d^{-1}t_2$ where d is the common 2-piece of (g,h). We know that $\phi(g)\omega(h,i) + \delta^2\eta(g,h,i)$ is equal to

$$\zeta(g,h,i) + \zeta(h,1,i) - \zeta(gh,1,i).$$

Using Proposition 4.2.5, (1) we see that

- $\zeta(g,h,i)$ is uniformly close to $\zeta(t_1^{-1}, t_2, i) + \zeta(d, d^{-1}t_2, i)$,
- $\zeta(h,1,i)$ is uniformly close to $\zeta(d^{-1}, t_2, i) + \zeta(t_2, 1, i)$ and
- $\zeta(gh,1,i)$ is uniformly close to $\zeta(t_1^{-1}, t_2, i) + \zeta(t_2, 1, i)$.

Combining these estimates we see that $\phi(g)\omega(h,i) + \delta^2\eta(g,h,i)$ is uniformly close to $\zeta(d, d^{-1}t_2, i) + \zeta(d^{-1}, t_2, i)$. $\square$

Proposition 4.2.9. *We have that $\phi(g)\omega(h,i) + \delta^2\eta(g,h,i)$ is uniformly close to*

$$\zeta(c, c^{-1}, c) + \zeta(c^{-1}, 1, c)$$

where c is the common 3-path of (g,h,i).

Proof. We consider the three different cases described in Subsection 4.1.5 of how three elements $g, h, i \in F$ can be aligned.

Case A : There are elements $t_1, \ldots, t_5$ such that $g = t_1t_2$, $h = t_2^{-1}t_3t_4$, $i = t_4^{-1}t_5$ as reduced words. Then the common 2-path of (g,h) is t_2. Hence $\phi(g)\omega(h,i) + \delta^2\eta(g,h,i)$ is uniformly close to

$$\zeta(t_2, t_2^{-1}t_3t_4, t_4^{-1}t_5) + \zeta(t_2^{-1}, t_3t_4, t_4^{-1}t_5).$$

Using Proposition 4.2.5, (2) for $u_1 = t_2^{-1}, u_2 = t_3, u_3 = t_4, u_4 = t_5$ we see that in this case $\phi(g)\omega(h,i) + \delta^2\eta(g,h,i)$ is uniformly bounded.

Case B : There are elements $t_1, \ldots, t_5$ such that $g = t_1 t_2 t_3$, $h = t_3^{-1} t_4$, $i = t_4^{-1} t_2^{-1} t_5$ as reduced words. Then the common 2-path of (g, h) is t_3. Hence $\phi(g)\omega(h, i) + \delta^2\eta(g, h, i)$ is uniformly close to

$$\zeta(t_3, t_3^{-1}t_4, t_4^{-1}t_2^{-1}t_5) + \zeta(t_3^{-1}, t_4, t_4^{-1}t_2^{-1}t_5).$$

Using Proposition 4.2.5, (2) for $u_1 = t_3^{-1}, u_2 = \emptyset, u_3 = t_4, u_4 = t_2^{-1}t_5$ we see that in this case, $\phi(g)\omega(h, i) + \delta^2\eta(g, h, i)$ is uniformly bounded.

Case C : There are elements $t_1, \ldots, t_4$ and c such that $g = t_1^{-1}ct_2$, $h = t_2^{-1}c^{-1}t_3$, $i = t_3^{-1}ct_4$ as reduced words. Then the common 2-path of (g, h) is ct_2. Hence $\phi(g)\omega(h, i) + \delta^2\eta(g, h, i)$ is uniformly close to

$$\zeta(ct_2, t_2^{-1}c^{-1}t_3, t_3^{-1}ct_4) + \zeta(t_2^{-1}c^{-1}, t_3, t_3^{-1}ct_4)$$

Using Proposition 4.2.5 (1) we see that

- $\zeta(ct_2, t_2^{-1}c^{-1}t_3, t_3^{-1}ct_4)$ is uniformly close to $\zeta(c, c^{-1}t_3, t_3^{-1}ct_4) + \zeta(t_2, t_2^{-1}c^{-1}t_3, t_3^{-1}ct_4)$ and
- $\zeta(t_2^{-1}c^{-1}, t_3, t_3^{-1}ct_4)$ is uniformly close to $\zeta(t_2^{-1}, c^{-1}t_3, t_3^{-1}ct_4) + \zeta(c^{-1}, t_3, t_3^{-1}ct_4)$.

Hence $\phi(g)\omega(h, i) + \delta^2\eta(g, h, i)$ is uniformly close to

$$\zeta(c, c^{-1}t_3, t_3^{-1}ct_4) + \zeta(c^{-1}, t_3, t_3^{-1}ct_4) + \Big(\zeta(t_2, t_2^{-1}c^{-1}t_3, t_3^{-1}ct_4) + \zeta(t_2^{-1}, c^{-1}t_3, t_3^{-1}ct_4)\Big).$$

Using Proposition 4.2.5 (2) for $u_1 = t_2^{-1}, u_2 = c^{-1}, u_3 = t_3, u_4 = ct_4$ we see that $\Big(\zeta(t_2, t_2^{-1}c^{-1}t_3, t_3^{-1}ct_4) + \zeta(t_2^{-1}, c^{-1}t_3, t_3^{-1}ct_4)\Big)$ is uniformly bounded. Using item (3a) of the same proposition for $u = c, v_1 = t_3^{-1}, v_2 = t_4$ we see that $\zeta(c, c^{-1}t_3, t_3^{-1}ct_4)$ is uniformly close to $\zeta(c, c^{-1}, c)$ and by item (3b) again for $u = c, v_1 = t_3^{-1}, v_2 = t_4$ we see that $\zeta(c^{-1}, t_3, t_3^{-1}ct_4)$ is uniformly close to $\zeta(c^{-1}, 1, c)$. Putting the above estimations together we see that $\phi(g)\omega(h, i) + \delta^2\eta(g, h, i)$ is uniformly close to $\zeta(c, c^{-1}, c) + \zeta(c^{-1}, 1, c)$.

□

Proposition 4.2.10. *The map $\theta\colon F \to \mathbb{R}$ defined by setting*

$$\theta\colon g \mapsto \zeta(g, g^{-1}, g) + \zeta(g^{-1}, 1, g)$$

is a symmetric quasimorphism.

Proof. We will first show the following claim:

Claim 4.2.11. *If $v, w \in F$ are such that vw is reduced then $\theta(vw)$ is uniformly close to $\theta(v) + \theta(w)$.*

Proof. Note that $\theta(vw) = \zeta(vw, w^{-1}v^{-1}, vw) + \zeta(w^{-1}v^{-1}, 1, vw)$. Using Proposition 4.2.5 (1) we see that $\theta(vw)$ is uniformly close to

$$\zeta(w, w^{-1}v^{-1}, vw) + \zeta(v, v^{-1}, vw) + \zeta(w^{-1}, v^{-1}, vw) + \zeta(v^{-1}, 1, vw).$$

By item (3) of the same proposition we see that

- $\zeta(w, w^{-1}v^{-1}, vw)$ is uniformly close to $\zeta(w, w^{-1}, w)$, for $u = w, v_1 = v, v_2 = \emptyset$,
- $\zeta(v, v^{-1}, vw)$ is uniformly close to $\zeta(v, v^{-1}, v)$, for $u = v, v_1 = \emptyset, v_2 = w$,
- $\zeta(w^{-1}, v^{-1}, vw)$ is uniformly close to $\zeta(w^{-1}, 1, w)$ for $u = w, v_1 = v, v_2 = \emptyset$ and
- $\zeta(v^{-1}, 1, vw)$ is uniformly close $\zeta(v^{-1}, 1, v)$ for $u = v, v_1 = \emptyset, v_2 = w$.

Putting things together we see that $\theta(vw)$ is uniformly close to

$$\Big(\zeta(v, v^{-1}, v) + \zeta(v^{-1}, 1, v)\Big) + \Big(\zeta(w, w^{-1}, w) + \zeta(w^{-1}, 1, w)\Big) = \theta(v) + \theta(w).$$

□

Claim 4.2.12. *The map $\theta\colon F \to \mathbb{R}$ is symmetric i.e. $\theta(g) = -\theta(g^{-1})$ for all $g \in F$.*

Proof. We first need two easy properties of ω. Note that ω is induced by a symmetric quasimorphism, say $\omega = \delta^1\rho$ for some quasimorphism $\rho\colon F \to \mathbb{R}$. We have that for all $u, v \in F$,

$$\omega(u, u^{-1}v) = \rho(u)+\rho(u^{-1}v)-\rho(v) = -\rho(u^{-1})-\rho(v)+\rho(u^{-1}v) = -\omega(u^{-1}, v). \quad (4.4)$$

and

$$\omega(u, v) = \rho(u) + \rho(v) - \rho(uv) = -\rho(u^{-1}) - \rho(v^{-1}) - \rho(v^{-1}u^{-1}) = -\omega(v^{-1}, u^{-1}). \quad (4.5)$$

Fix $g \in F$ such that $\Delta(g) = (g_1, \ldots, g_k)$. Recall that in this case $\Delta(g^{-1}) = (g_k^{-1}, \ldots, g_1^{-1})$. Then

- $\zeta(g, g^{-1}, g) = \sum_{j=1}^{k} \phi(g_j)\omega(g_j^{-1} \cdots g_1^{-1}, g) = \sum_{j=1}^{k} \phi(g_j)\omega(g_1 \cdots g_j, g_{j+1} \cdots g_k)$ using (4.4) for $u = g_j^{-1} \cdots g_1^{-1}$ and $v = g_{j+1} \cdots g_k$. Similarly we see that

- $\zeta(g^{-1}, 1, g) = \sum_{j=1}^{k} \phi(g_j^{-1})\omega(g_1 \cdots g_{j-1}, g_j \cdots g_k) = -\sum_{j=1}^{k} \phi(g_j)\omega(g_1 \cdots g_{j-1}, g_j \cdots g_k)$ using that ϕ is symmetric,

- $\zeta(g^{-1}, g, g^{-1}) = \sum_{j=1}^{k} \phi(g_j^{-1})\omega(g_k^{-1} \cdots g_j^{-1}, g_{j-1}^{-1} \cdots g_1^{-1}) = \sum_{j=1}^{k} \phi(g_j)\omega(g_1 \cdots g_{j-1}, g_j \cdots g_k)$ where we used that ϕ is symmetric and (4.5) and

- $\zeta(g, 1, g^{-1}) = \sum_{j=1}^{k} \phi(g_j)\omega(g_k^{-1} \cdots g_j^{-1}, g_{j-1}^{-1} \cdots g_1^{-1}) = -\sum_{j=1}^{k} \phi(g_j)\omega(g_1 \cdots g_j, g_{j+1} \cdots g_k)$, where we used once more (4.5).

We hence see that $\theta(g) + \theta(g^{-1}) = \zeta(g, g^{-1}, g) + \zeta(g^{-1}, 1, g) + \zeta(g^{-1}, g, g^{-1}) + \zeta(g, 1, g^{-1}) = 0$ and θ is symmetric. □

We can now prove that θ is a quasimorphism. Let $g, h \in F$ and suppose that d is the common 2-path of (g, h) i.e. $g = t_1^{-1}d$, $h = d^{-1}t_2$ as reduced words for some appropriate $t_1, t_2 \in F$. Then, by Claim 4.2.11 we have that $\theta(g) + \theta(h)$ is uniformly close to

$$\theta(t_1^{-1}) + \theta(d) + \theta(d^{-1}) + \theta(t_2)$$

and by Claim 4.2.12, $\theta(g) + \theta(h)$ is uniformly close to $\theta(t_1^{-1}) + \theta(t_2)$. By Claim 4.2.11 again, $\theta(t_1^{-1}) + \theta(t_2)$ is uniformly close to $\theta(t_1^{-1}t_2) = \theta(gh)$. Hence $\theta(g) + \theta(h)$ is uniformly close to $\theta(gh)$ and hence θ is a quasimorphism. □

We will need the following Lemma:

Lemma 4.2.13. *Suppose $\rho \colon F \to \mathbb{R}$ is a symmetric quasimorphism. Define $\kappa \in C^2(F, \mathbb{R})$ by $\kappa(g, h) = \rho(d)$ where d is the common 2-path of (g, h). Then $\delta^2\kappa(g, h, i)$ is uniformly close to $-2\rho(c)$ where c is the common 3-path of (g, h, i).*

Proof. We have to evaluate

$$\delta^2\kappa(g, h, i) = \kappa(h, i) - \kappa(gh, i) + \kappa(g, hi) - \kappa(g, h).$$

For what follows we will use the different cases of how g, h and i can be aligned in the Cayley graph of F as seen in Figure 4.3.

1. (see Figure 4.3a): In this case there are elements $t_1, \dots, t_5$ such that $g = t_1 t_2$, $h = t_2^{-1} t_3 t_4$, $i = t_4^{-1} t_5$ as reduced words. It follows that

   - t_4 is the common 2-path of (h, i),
   - t_4 is the common 2-path of (gh, i),
   - t_2 is the common 2-path of (g, hi) and
   - t_2 is the common 2-path of (g, h).

   Hence $\delta^2 \kappa(g, h, i) = \rho(t_4) - \rho(t_4) + \rho(t_2) - \rho(t_2) = 0$.

2. (see Figure 4.3b): In this case there are elements $t_1, \dots, t_5$ such that $g = t_1 t_2 t_3$, $h = t_3^{-1} t_4$, $i = t_4^{-1} t_2^{-1} t_5$ as reduced words. It follows that

   - t_4 is the common 2-path of (h, i),
   - $t_4 t_2$ is the common 2-path of (gh, i),
   - $t_2 t_3$ is the common 2-path of (g, hi) and
   - t_3 is the common 2-path of (g, h).

   Hence $\delta^2 \kappa(g, h, i) = \rho(t_4) - \rho(t_4 t_2) + \rho(t_2 t_3) - \rho(t_3)$ which is uniformly bounded as ρ is a quasimorphism.

3. (see Figure 4.3c): In this case there are elements $t_1, \dots, t_4$ and c such that $g = t_1^{-1} c t_2$, $h = t_2^{-1} c^{-1} t_3$, $i = t_3^{-1} c t_4$ as reduced words. It follows that

   - $c^{-1} t_3$ is the common 2-path of (h, i),
   - t_3 is the common 2-path of (gh, i),
   - t_2 is the common 2-path of (g, hi) and
   - $c t_2$ is the common 2-path of (g, h).

   Hence $\delta^2 \kappa(g, h, i) = \rho(c^{-1} t_3) - \rho(t_3) + \rho(t_2) - \rho(c t_2)$ which is uniformly close to $-2\rho(c)$. This shows Lemma 4.2.13.

□

Finally, we can prove Theorem H. By Proposition 4.2.9, $\phi(g)\omega(h,i)+\delta^2\eta(g,h,i)$ is uniformly close to $\zeta(c,c^{-1},c)+\zeta(c^{-1},1,c)=\theta(c)$ where c is the common 3-path of (g,h,i) and $\theta\colon F\to\mathbb{R}$ is like in Proposition 4.2.10. Define $\gamma\in C^2(F,\mathbb{R})$ via $\gamma(g,h)=\theta(d)/2$ where d is the common 2-path of (g,h). Observe that $\rho\colon g\mapsto\theta(g)/2$ is a symmetric quasimorphism by Proposition 4.2.10. Using Lemma 4.2.13, we see that $\delta^2\gamma(g,h,i)$ is uniformly close to $-\theta(c)$ where c is the common 3-path of (g,h,i). Hence $\phi(g)\omega(h,i)+\delta^2\eta(g,h,i)+\delta^2\gamma(g,h,i)$ is uniformly bounded.

4.2.4 Proof of Theorems C and D

Here we will prove Theorems C and D by providing an explicit bounded primitive for the respective cup products.

Theorem D. *Let Δ be a decomposition of F, let ϕ be a Δ-decomposable quasimorphism and let ψ be Δ-continuous. Then $[\delta^1\phi]\smile[\delta^1\psi]\in\mathrm{H}^4_b(F,\mathbb{R})$ is trivial. The bounded primitive is given by β, as in Theorem H for $\omega=\delta^1\psi$.*

Proof. By Theorem H we know that β defined by setting $\beta\colon(g,h,i)\mapsto\phi(g)\delta^1\psi(h,i)+\delta^2\eta(g,h,i)+\delta^2\gamma(g,h,i)$ is bounded, as $\delta^1\psi(h,i)$ is a symmetric Δ-continuous cocycle. Then we calculate

$$\delta^3\beta(g,h,i,j)=\delta^1\phi(g,h)\smile\delta^1\psi(i,j).$$

Hence β is a bounded primitive for the cup product. □

Finally, we can prove Theorem C.

Theorem C. *Let $\phi,\psi\colon F\to\mathbb{R}$ be two quasimorphisms on a non-abelian free group F where each of ϕ and ψ is either Brooks counting quasimorphisms on a non self-overlapping word or quasimorphisms in the sense of Rolli. Then $[\delta^1\phi]\smile[\delta^1\psi]\in\mathrm{H}^4_b(F,\mathbb{R})$ is trivial.*

Proof. First suppose that both ϕ and ψ are Brooks quasimorphisms. Suppose that ϕ is counting the non-overlapping word $w\in F$. Let Δ_w be the decomposition described in Example 4.1.3. By Example 2.2.7, we have that ϕ is Δ_w-decomposable. Moreover, by Proposition 4.1.12, ψ is Δ_w-continuous. We conclude by Theorem D.

If not both ϕ and ψ are Brooks quasimorphisms then assume without loss of generality that ϕ is a quasimorphism in the sense of Rolli and ψ is either a Brooks quasimorphism or a quasimorphism in the sense of Rolli. Let Δ_{rolli} be the decomposition described in Example 4.1.4. Note that ϕ is Δ_{rolli}-decomposable. If ψ is a quasimorphism in the sense of Rolli, then ψ is Δ_{rolli}-decomposable and hence Δ_{rolli}-continuous by Proposition 4.1.12. If ψ is a Brooks quasimorphism then by the same proposition we see that ψ is also Δ_{rolli}-continuous. Again we may conclude by applying Theorem D. □

Chapter 5

Gaps in scl for RAAGs

It is a phenomenon that many "negatively curved" groups have a *gap* in stable commutator length, i.e. there is a constant $C > 0$ such that every element $g \in G'$ either satisfies $\mathrm{scl}(g) = 0$ or satisfies $\mathrm{scl}(g) \geq C$; see Subsection 2.3.2.

Some groups even satisfy the stronger property that in addition the only element which satisfies $\mathrm{scl}(g) = 0$ is the identity. Such a gap is interesting since it is inherited by subgroups. Recall from Subsection 2.3.2 that both such gaps are necessarily bounded above by 1/2. The aim of this chapter is to prove that this gap is indeed exactly 1/2 for certain amalgamated free products and right-angled Artin groups.

A common way of establishing gaps in scl is by constructing *quasimorphisms* and using *Bavard's Duality Theorem* (see Theorem 2.3.2 and [Bav91]): For an element $g \in G'$,

$$\mathrm{scl}(g) = \sup_{\bar{\phi} \in \mathcal{Q}(G)} \frac{\bar{\phi}(g)}{2D(\bar{\phi})}$$

where $\mathcal{Q}(G)$ is the space of *homogeneous quasimorphisms* and $D(\bar{\phi})$ is the *defect of* $\bar{\phi}$; see Subsection 2.2.5 for the definitions and the precise statement.

In the first part of this chapter, we will construct a family of extremal quasimorphisms on non-abelian free groups. Let $\mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle$ be the free group on generators $\mathtt{a}$ and $\mathtt{b}$ and let $w \in \mathbb{F}_2$ be such that it does not conjugate into $\langle \mathtt{a} \rangle$ or $\langle \mathtt{b} \rangle$. Then we will construct a homogeneous quasimorphism $\bar{\phi}$ such that $\bar{\phi}(w) \geq 1$ and $D(\bar{\phi}) \leq 1$. This realises the well-known gap of 1/2 in the case of non-abelian free groups. Our approach is as follows: instead of constructing more complicated quasimorphisms $\bar{\phi}$ we first "simplify" the element w.

This simplification is formalised by functions $\Phi\colon G \to \mathcal{A} \subset \mathbb{F}_2$, called *letter-quasimorphisms*; see Definition 5.2.1. Here $\mathcal{A}$ denotes the set of *alternating words* in $\mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle$ with the generators $\mathtt{a}$ and $\mathtt{b}$. These are words where each letter alternates between $\{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\{\mathtt{b}, \mathtt{b}^{-1}\}$. Letter-quasimorphisms are a special case of quasimorphisms between arbitrary groups defined by Hartnick–Schweitzer [HS16]. After this simplification, the extremal quasimorphisms on G are obtained by pulling back most basic quasimorphisms $\mathbb{F}_2 \to \mathbb{R}$ via such letter-quasimorphisms $G \to \mathcal{A} \subset \mathbb{F}_2$. We further deduce that such quasimorphisms are induced by a circle action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ by examining the defect and using Theorem 2.2.4 due to Ghys; see also [Ghy87]. We show:

Theorem E. *Let G be a group, $g \in G$ and suppose that there is a letter-quasimorphism $\Phi\colon G \to \mathcal{A}$ such that $\Phi(g)$ is non-trivial and $\Phi(g^n) = \Phi(g)^n$ for all $n \in \mathbb{N}$ Then there is an explicit homogeneous quasimorphism $\bar{\phi}\colon G \to \mathbb{R}$ such that $\bar{\phi}(g) \geq 1$ and $D(\bar{\phi}) \leq 1$.*

If G is countable then there is an action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1 \bar{\phi}] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}_b^2(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.

By Bavard's Duality Theorem it is immediate that if such an element g additionally lies in G', then $\mathrm{scl}(g) \geq 1/2$. We state Theorem E separately as it may also be applied in other cases than the ones presented in this chapter; see Remark 5.4.3. Many groups G have the property that for any element $g \in G'$ there is a letter-quasimorphism $\Phi_g\colon G \to \mathcal{A}$ such that $\Phi_g(g^n) = \Phi_g(g)^n$ where $\Phi_g(g) \in \mathcal{A}$ is non-trivial. We will see that residually free groups and right-angled Artin groups have this property. Note the similarities of this property with being *residually free*; see Remark 5.2.8.

In the second part of this chapter we apply Theorem E to amalgamated free products using left-orders. A subgroup $H < G$ is called *left-relatively convex* if there is an order on the left cosets G/H which is invariant under left multiplication by G. We will construct letter-quasimorphisms $G \to \mathcal{A} \subset \mathbb{F}_2$ using the sign of these orders. We deduce:

Theorem F. *Let A, B, C be groups, $\kappa_A \colon C \hookrightarrow A$ and $\kappa_B \colon C \hookrightarrow B$ injections and suppose both $\kappa_A(C) < A$ and $\kappa_B(C) < B$ are left-relatively convex. If $g \in A \star_C B$ does not conjugate into one of the factors then there is a homogeneous quasimorphism $\bar{\phi} \colon A \star_C B \to \mathbb{R}$ such that $\bar{\phi}(g) \geq 1$ and $D(\bar{\phi}) \leq 1$. If G is countable then there is an action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1 \bar{\phi}] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}_b^2(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.*

It is possible to generalise Theorem F to graphs of groups; see Remark 5.4.3. Again by Bavard's Duality Theorem we infer that any such g which also lies in the commutator subgroup satisfies $\mathrm{scl}(g) \geq 1/2$. We apply this to right-angled Artin groups using the work of [ADS15]. This way we prove:

Theorem G. *Every non-trivial element $g \in G'$ in the commutator subgroup of a right-angled Artin group G satisfies $\mathrm{scl}(g) \geq 1/2$. This bound is sharp.*

This is an improvement of the bound previously found in [FFT16] and [FST17] who deduced a general bound of 1/24 and a bound of 1/20 if the right-angled Artin group is two dimensional. Every subgroup of a right-angled Artin group will inherit this bound. Such groups are now known to be an extremely rich class, following the theory of special cube complexes. See [Wis09], [HW08], [Ago13], [Bri13] and [Bri17]. Stable commutator length may serve as an invariant to distinguish virtually special from special cube complexes.

Properties of the constructed quasimorphisms

We collect some properties of the quasimorphisms constructed in this chapter.

- The quasimorphisms are induced by circle actions $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ even though we do not construct the explicit action ρ. In particular, for every $e \neq g \in F'$ where F is a non-abelian free group and $\mathrm{scl}(g) = 1/2$ there is an *extremal* quasimorphism $\bar{\phi} \colon F \to \mathbb{R}$ induced by a circle action. It is unknown if for an arbitrary element $g \in F'$ there is an action of F on the circle such that the induced quasimorphism is extremal with respect to g.

- There are relatively few quasimorphisms needed to obtain the $1/2$ bound in Theorem G. Let G be a right-angled Artin group. Analysis of the constructions show that there is a sequence $\mathcal{S}_N \subset \mathcal{Q}(G)$ of nested sets of homogeneous quasimorphisms such that for every non-trivial cyclically reduced element g of length less than N there is some $\bar{\phi} \in \mathcal{S}_N$ such that $\bar{\phi}(g) \geq 1$ and $D(\bar{\phi}) \leq 1$. We see that $|\mathcal{S}_N| = O(N)$ and the rate-constant only depends on the number of generators of the right-angled Artin group.

- We obtain gap results even for elements which are not in the commutator subgroup. This suggests that it may be interesting to use Bavard's Dualtiy Theorem as a generalisation of stable commutator length to an invariant of general group elements $g \in G$. That is to study the supremum of $\bar{\phi}(g)/2$ where $\bar{\phi}$ ranges over all homogeneous quasimorphisms with $D(\bar{\phi}) = 1$ which vanish or are bounded on a fixed generating set. In [CW11] the authors studied this supremum over all homogeneous quasimorphisms induced by circle actions. They could prove that this supremum has certain qualitative similarities to the experimental values observed for scl. This includes the experimental phenomenon that values with low denominators appear more frequently in scl.

Organisation

In Section 5.1 we introduce *letter-thin triples* which are a special type of triples (x_1, x_2, x_3) of alternating elements $x_1, x_2, x_3 \in \mathcal{A}$. These will be crucial in estimating the defect of the quasimorphisms constructed in this chapter. We will define maps $\alpha, \beta \colon \mathcal{A} \to \mathcal{A}$, which we show to respect letter-thin triples in Lemma 5.1.14. In Section 5.2 we define and study *letter-quasimorphisms* which are maps from arbitrary groups to alternating words of the free group. We deduce Theorem E which serves as a criterion for scl-gaps of $1/2$ using these letter-quasimorphisms. Section 5.3 recalls some results of [ADS15] on left relatively convex subgroups and orders on groups. Using the sign of these orders we are able to deduce $1/2$ gaps for amalgamated free products in Section 5.4; see Theorem F. We show the $1/2$ gaps for right-angled Artin groups in Section 5.5; see Theorem G.

5.1 Letter-Thin Triples and the Maps α and β

The set of alternating words $\mathcal{A} \subset \mathbb{F}_2$ is the set of all words in the letters $\mathtt{a}$ and $\mathtt{b}$ where the letters alternate between $\{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\{\mathtt{b}, \mathtt{b}^{-1}\}$. For example, $\mathtt{aba}^{-1}\mathtt{b}^{-1}$ is an alternating word but $\mathtt{abba}^{-1}\mathtt{b}^{-1}\mathtt{b}^{-1}$ is not. We will define maps $\alpha, \beta \colon \mathcal{A} \to \mathcal{A}$ and develop their basic properties in Subsection 5.1.1. We also define a version of these maps on $\bar{\mathcal{A}}_0$, which are conjugacy classes of *even*-length words of $\mathcal{A}$ to understand how α, β behave on powers; see Proposition 5.1.9. In Subsection 5.1.2 we define certain triples (x_1, x_2, x_3) where $x_1, x_2, x_3 \in \mathcal{A}$ called *letter-thin triples*. We think of them as the sides of (thin) triangles; see Figure 5.2. Note that such triples are not triangles in the usual sense, i.e. the sides x_1, x_2, x_3 do *not* correspond to the geodesics between three points in some metric space like a Cayley graph. Letter-thin triples will be crucial in estimating the defect of the quasimorphisms we construct in this paper. We will see that α and β map letter-thin triples to letter-thin triples in Lemma 5.1.14, which is the main technical result of this paper. In Subsection 5.1.3 we see that basic Brooks quasimorphisms and homomorphisms behave well on letter-thin triples. We usually prove the properties we state for α, β just for α and note that all properties may be deduced analogously for β by interchanging $\mathtt{a}$ and $\mathtt{b}$; see Proposition 5.1.4, (2).

5.1.1 The Maps α and β, Definition and Properties

We will describe two maps $\alpha, \beta \colon \mathcal{A} \to \mathcal{A}$ sending alternating words to alternating words. Define $\mathcal{S}^+_{\mathtt{a}}, \mathcal{S}^-_{\mathtt{a}} \subset \mathcal{A}$ as

$$\begin{aligned} \mathcal{S}^+_{\mathtt{a}} &= \{\mathtt{a}\mathtt{y}_1\mathtt{a} \cdots \mathtt{a}\mathtt{y}_l\mathtt{a} \mid \mathtt{y}_i \in \{\mathtt{b}, \mathtt{b}^{-1}\}, l \in \mathbb{N}\} \\ \mathcal{S}^-_{\mathtt{a}} &= \{\mathtt{a}^{-1}\mathtt{y}_1\mathtt{a}^{-1} \cdots \mathtt{a}^{-1}\mathtt{y}_l\mathtt{a}^{-1} \mid \mathtt{y}_i \in \{\mathtt{b}, \mathtt{b}^{-1}\}, l \in \mathbb{N}\} \end{aligned}$$

that is, $\mathcal{S}^+_{\mathtt{a}}$ is the set of alternating words which start and end in $\mathtt{a}$ and don't contain the letter $\mathtt{a}^{-1}$ and $\mathcal{S}^-_{\mathtt{a}}$ is the set of alternating words which start and end in $\mathtt{a}^{-1}$ and don't contain the letter $\mathtt{a}$. Note that we assume $0 \in \mathbb{N}$, i.e. $\mathtt{a} \in \mathcal{S}^+_{\mathtt{a}}$ and $\mathtt{a}^{-1} \in \mathcal{S}^-_{\mathtt{a}}$.

Analogously we define the sets $\mathcal{S}_{\mathtt{b}}^{+} \subset \mathcal{A}$ and $\mathcal{S}_{\mathtt{b}}^{-} \subset \mathcal{A}$ as

$$\mathcal{S}_{\mathtt{b}}^{+} = \{\mathtt{b}\mathtt{x}_1\mathtt{b}\cdots\mathtt{b}\mathtt{x}_l\mathtt{b} \mid \mathtt{x}_i \in \{\mathtt{a},\mathtt{a}^{-1}\}, l \in \mathbb{N}\}$$
$$\mathcal{S}_{\mathtt{b}}^{-} = \{\mathtt{b}^{-1}\mathtt{x}_1\mathtt{b}^{-1}\cdots\mathtt{b}^{-1}\mathtt{x}_l\mathtt{b}^{-1} \mid \mathtt{x}_i \in \{\mathtt{a},\mathtt{a}^{-1}\}, l \in \mathbb{N}\}$$

and observe that $\mathtt{b} \in \mathcal{S}_{\mathtt{b}}^{+}$ and $\mathtt{b}^{-1} \in \mathcal{S}_{\mathtt{b}}^{-}$.

We will decompose arbitrary words $w \in \mathcal{A}$ as a *unique* product of elements in $\{\mathtt{b},\mathtt{b}^{-1}\}$ and $\mathcal{S}_{\mathtt{a}}^{+} \cup \mathcal{S}_{\mathtt{a}}^{-}$:

Proposition 5.1.1. *Let $w \in \mathcal{A}$ be an alternating word. Then there are $\mathtt{y}_0,\ldots,\mathtt{y}_l$ and $s_1,\ldots,s_l$ such that*

$$w = \mathtt{y}_0 s_1 \mathtt{y}_1 s_2 \cdots \mathtt{y}_{l-1} s_l \mathtt{y}_l$$

where $\mathtt{y}_i \in \{\mathtt{b},\mathtt{b}^{-1}\}$ except that $\mathtt{y}_0$ and/or $\mathtt{y}_l$ may be empty and $s_i \in \mathcal{S}_{\mathtt{a}}^{+} \cup \mathcal{S}_{\mathtt{a}}^{-}$. Moreover, s_i alternates between $\mathcal{S}_{\mathtt{a}}^{+}$ and $\mathcal{S}_{\mathtt{a}}^{-}$, i.e. there is no $i \in \{1,\ldots,l-1\}$ such that $s_i, s_{i+1} \in \mathcal{S}_{\mathtt{a}}^{+}$ or $s_i, s_{i+1} \in \mathcal{S}_{\mathtt{a}}^{-}$. This expression is unique.

We will call this way of writing w the *$\mathtt{a}$-decomposition of w*. Analogously, we may also write $w \in \mathcal{A}$ as

$$w = \mathtt{x}_0 t_1 \mathtt{x}_1 t_2 \cdots \mathtt{x}_{l-1} t_l \mathtt{x}_l$$

(possibly with a different l), where $\mathtt{x}_i \in \{\mathtt{a},\mathtt{a}^{-1}\}$ except that $\mathtt{x}_0$ and / or $\mathtt{x}_l$ may be empty and $t_i \in \mathcal{S}_{\mathtt{b}}^{+} \cup \mathcal{S}_{\mathtt{b}}^{-}$ where t_i alternate between $\mathcal{S}_{\mathtt{b}}^{+}$ and $\mathcal{S}_{\mathtt{b}}^{-}$. We will call this way of writing w the *$\mathtt{b}$-decomposition of w*.

Proof. (of Proposition 5.1.1) Let $w \in \mathcal{A}$ be an alternating word. Since $\mathtt{a} \in \mathcal{S}_{\mathtt{a}}^{+}$ and $\mathtt{a}^{-1} \in \mathcal{S}_{\mathtt{a}}^{-}$, we may always find some $s_i \in \mathcal{S}_{\mathtt{a}}^{+} \cup \mathcal{S}_{\mathtt{a}}^{-}$ and some $\mathtt{y}_i \in \{\mathtt{b},\mathtt{b}^{-1}\}$ such that

$$w = \mathtt{y}_0 s_1 \mathtt{y}_1 s_2 \cdots \mathtt{y}_{n-1} s_n \mathtt{y}_n$$

with possibly $\mathtt{y}_n$ and / or $\mathtt{y}_0$ empty.

Now let m be the minimal n of all such products representing w i.e.

$$w = \mathtt{y}_0 s_1 \mathtt{y}_1 s_2 \cdots \mathtt{y}_{m-1} s_m \mathtt{y}_m.$$

Suppose there is an $i \in \{1,\ldots,m-1\}$ such that $s_i, s_{i+1} \in \mathcal{S}_{\mathtt{a}}^{+}$ (resp. $s_i, s_{i+1} \in \mathcal{S}_{\mathtt{a}}^{-}$). Set $s' = s_i \mathtt{y}_i s_{i+1}$ and note that $s' \in \mathcal{S}_{\mathtt{a}}^{+}$ (resp. $s' \in \mathcal{S}_{\mathtt{a}}^{-}$). Then

$$w = \mathtt{y}_0 s_1 \mathtt{y}_1 s_2 \cdots \mathtt{y}_{i-1} s' \mathtt{y}_{i+1} \cdots \mathtt{y}_{m-1} s_m \mathtt{y}_m$$

which would contradict the minimality of m. Hence all s_i alternate between $\mathcal{S}_{\mathtt{a}}^{+}$ and $\mathcal{S}_{\mathtt{a}}^{-}$. By comparing two such expressions we see that such an expression is further unique. □

Definition 5.1.2. Let $w \in \mathcal{A}$ and let $w = \mathtt{y}_0 s_1 \cdots \mathtt{y}_{l-1} s_l \mathtt{y}_l$ be the $\mathtt{a}$-decomposition of w. Then $\alpha \colon \mathcal{A} \to \mathcal{A}$ is defined via

$$\alpha \colon w \mapsto \mathtt{y}_0 \mathtt{x}_1 \mathtt{y}_1 \mathtt{x}_2 \cdots \mathtt{y}_{l-1} \mathtt{x}_l \mathtt{y}_l$$

with $\mathtt{x}_i = \mathtt{a}$ if $s_i \in \mathcal{S}_{\mathtt{a}}^{+}$ and $\mathtt{x}_i = \mathtt{a}^{-1}$ if $s_i \in \mathcal{S}_{\mathtt{a}}^{-}$.

Analogously suppose that $w = \mathtt{x}_0 t_1 \mathtt{x}_1 t_2 \cdots \mathtt{x}_{l-1} t_l \mathtt{x}_l$ is the $\mathtt{b}$-decomposition of w, with l possibly different from above. We define the map $\beta \colon \mathcal{A} \to \mathcal{A}$ via

$$\beta \colon w \mapsto \mathtt{x}_0 \mathtt{y}_1 \mathtt{x}_1 \mathtt{y}_2 \cdots \mathtt{x}_{l-1} \mathtt{y}_l \mathtt{x}_l$$

with $\mathtt{y}_i = \mathtt{b}$ if $t_i \in \mathcal{S}_{\mathtt{b}}^{+}$ and $\mathtt{y}_i = \mathtt{b}^{-1}$ if $t_i \in \mathcal{S}_{\mathtt{b}}^{-}$.

Example 5.1.3. Let $w = \mathtt{bab}^{-1}\mathtt{abab}^{-1}\mathtt{a}^{-1}\mathtt{ba}^{-1}\mathtt{baba}^{-1}$. Then the $\mathtt{a}$-decomposition of w is

$$w = \mathtt{b} s_1 \mathtt{b}^{-1} s_2 \mathtt{b} s_3 \mathtt{b} s_4$$

where $s_1 = \mathtt{ab}^{-1}\mathtt{aba} \in \mathcal{S}_{\mathtt{a}}^{+}$, $s_2 = \mathtt{a}^{-1}\mathtt{ba}^{-1} \in \mathcal{S}_{\mathtt{a}}^{-}$, $s_3 = \mathtt{a} \in \mathcal{S}_{\mathtt{a}}^{+}$ and $s_4 = \mathtt{a}^{-1} \in \mathcal{S}_{\mathtt{a}}^{+}$. Hence

$$\alpha(w) = \mathtt{bab}^{-1}\mathtt{a}^{-1}\mathtt{baba}^{-1}.$$

Observe that then $\alpha(\alpha(w)) = \alpha(w)$. The $\mathtt{b}$-decomposition of $\alpha(w)$ is

$$\alpha(w) = t_1 \mathtt{a} t_2 \mathtt{a}^{-1} t_3 \mathtt{a}^{-1}$$

where $t_1 = \mathtt{b} \in \mathcal{S}_{\mathtt{b}}^{+}$, $t_2 = \mathtt{b}^{-1} \in \mathcal{S}_{\mathtt{b}}^{-}$ and $t_3 = \mathtt{bab} \in \mathcal{S}_{\mathtt{b}}^{+}$. Hence

$$\beta(\alpha(w)) = \mathtt{bab}^{-1}\mathtt{a}^{-1}\mathtt{ba}^{-1}$$

and similarly, we may see that $\alpha(\beta(\alpha(w))) = \mathtt{bab}^{-1}\mathtt{a}^{-1} = [\mathtt{b}, \mathtt{a}]$. Then both $\alpha([\mathtt{b}, \mathtt{a}]) = [\mathtt{b}, \mathtt{a}]$ and $\beta([\mathtt{b}, \mathtt{a}]) = [\mathtt{b}, \mathtt{a}]$. We will formalise and use this behaviour later; see Proposition 5.1.4 and Proposition 5.1.8.

The images of α and β are obviously contained in the set of alternating words. Moreover, as the s_i in the previous definition all alternate between $\mathcal{S}_{\mathtt{a}}^+$ and $\mathcal{S}_{\mathtt{a}}^-$, none of the consecutive $\mathtt{x}_i$ have the same sign in the image of α and no consecutive $\mathtt{y}_i$ have the same sign in the image of β.

Proposition 5.1.4. *The maps $\alpha, \beta \colon \mathcal{A} \to \mathcal{A}$ have the following properties:*

1. *For every $w \in \mathcal{A}$, $\alpha(w^{-1}) = \alpha(w)^{-1}$ and $\beta(w^{-1}) = \beta(w)^{-1}$*
2. *$\psi \circ \alpha = \beta \circ \psi$ and $\psi \circ \beta = \alpha \circ \psi$, where $\psi \colon \mathbb{F}_2 \to \mathbb{F}_2$ is the automorphism defined via $\psi \colon \mathtt{a} \mapsto \mathtt{b}, \mathtt{b} \mapsto \mathtt{a}$.*
3. *For any $w \in \mathcal{A}$, $\alpha(\alpha(w)) = \alpha(w)$. Moreover, $|\alpha(w)| \leq |w|$ with equality if and only if $\alpha(w) = w$. The analogous statement holds for β.*
4. *Let $v_1 \mathtt{x} v_2$ be an alternating word with $v_1, v_2 \in \mathcal{A}$ and $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$. Then $\alpha(v_1 \mathtt{x} v_2)$ is equal in $\mathbb{F}_2$ to the element represented by the non-reduced word $\alpha(v_1 \mathtt{x})\mathtt{x}^{-1}\alpha(\mathtt{x} v_2)$. The analogous statement holds for β.*

Proof. To see (1), note that if $w = \mathtt{y}_0 s_1 \mathtt{y}_1 \cdots \mathtt{y}_{l-1} s_l \mathtt{y}_l$ is the $\mathtt{a}$-decomposition of w, then

$$\mathtt{y}_l^{-1} s_l^{-1} \mathtt{y}_{l-1}^{-1} \cdots \mathtt{y}_1^{-1} s_1^{-1} \mathtt{y}_0^{-1}$$

is the $\mathtt{a}$-decomposition of w^{-1}. As $s_i^{-1} \in \mathcal{S}_{\mathtt{a}}^+$ if and only if $s_i \in \mathcal{S}_{\mathtt{a}}^-$ and $s_i^{-1} \in \mathcal{S}_{\mathtt{a}}^-$ if and only if $s_i \in \mathcal{S}_{\mathtt{a}}^+$ we can conclude that $\alpha(w^{-1}) = \alpha(w)^{-1}$. The analogous argument holds for β.

Point (2) is evident from the symmetric way α and β have been defined. To see (3), note that α replaces each of the subwords s_i by letters $\mathtt{a}$ or $\mathtt{a}^{-1}$. These have size strictly less than $|s_i|$ unless s_i is the letter $\mathtt{a}$ or $\mathtt{a}^{-1}$ already. This shows $|\alpha(w)| \leq |w|$ with equality only if $\alpha(w) = w$ and it also shows that $\alpha \circ \alpha = \alpha$.

For (4) suppose that the $\mathtt{a}$-decomposition of $v_1\mathtt{x}$ is $\mathtt{y}_0^1 s_1^1 \mathtt{y}_1^1 \cdots \mathtt{y}_{l_1-1}^1 s_{l_1}^1$ and the $\mathtt{a}$-decomposition of $\mathtt{x}v_2$ is $s_1^2 \mathtt{y}_1^2 \cdots \mathtt{y}_{l_1-1}^1 s_{l_2}^2 \mathtt{y}_{l_2}^2$. Both, $s_{l_1}^1$ and s_1^2 lie in the same set $S_{\mathtt{a}}^+$ or $S_{\mathtt{a}}^-$ depending if $\mathtt{x} = \mathtt{a}$ or $\mathtt{x} = \mathtt{a}^{-1}$. Without loss of generality assume that $\mathtt{x} = \mathtt{a}$. The $\mathtt{a}$-decomposition of $v_1\mathtt{x}v_2$ may be seen to be $\mathtt{y}_0^1 s_1^1 \mathtt{y}_1^1 \cdots \mathtt{y}_{l_1-1}^1 s \mathtt{y}_1^2 \cdots \mathtt{y}_{l_2-1}^2 s_{l_2}^2 \mathtt{y}_{l_2}^2$ where $s \in S_{\mathtt{a}}^+$ is equal to $s_{l_1}^1 \mathtt{a}^{-1} s_1^2$ in $\mathbb{F}_2$. Hence $\alpha(v_1\mathtt{a}) = \mathtt{y}_0^1 \mathtt{x}_1^1 \mathtt{y}_1^1 \cdots \mathtt{y}_{l_1-1}^1 \mathtt{a}$, $\alpha(\mathtt{a}v_2) = \mathtt{a}\mathtt{y}_1^2 \cdots \mathtt{y}_{l_2-1}^2 \mathtt{x}_{l_2}^2 \mathtt{y}_{l_2}^2$ and

$$\alpha(v_1\mathtt{x}v_2) = \mathtt{y}_0^1 \mathtt{x}_1^1 \mathtt{y}_1^1 \cdots \mathtt{y}_{l_1-1}^1 \mathtt{a} \mathtt{y}_1^2 \cdots \mathtt{y}_{l_2-1}^2 \mathtt{x}_{l_2}^2 \mathtt{y}_{l_2}^2.$$

Comparing terms finishes the proposition. □

To study how the maps $\alpha, \beta \colon \mathcal{A} \to \mathcal{A}$ behave on powers of elements we need to define a version of them on conjugacy classes. Let $\bar{\mathcal{A}}_0$ be the set conjugacy classes of even length alternating words. Note that then necessarily every two representatives $w_1, w_2 \in \mathcal{A}$ of the same conjugacy class in $\bar{\mathcal{A}}_0$ are equal up to cyclic permutation of the letters. This is, there are elements $v_1, v_2 \in \mathcal{A}$ such that $w_1 = v_1 v_2$ and $w_2 = v_2 v_1$ as reduced words. Hence every representative $v \in \mathcal{A}$ of an element in $\bar{\mathcal{A}}_0$ is automatically reduced.

Remark 5.1.5. Every reduced representative $w \in \mathcal{A}$ of a class in $\bar{\mathcal{A}}_0$ has the same length. Every homogeneous quasimorphism $\bar{\phi} \colon \mathbb{F}_2 \to \mathbb{R}$ depends only on conjugacy classes and hence induces a well-defined map $\bar{\phi} \colon \bar{\mathcal{A}}_0 \to \mathbb{R}$. We say that an element $[w] \in \bar{\mathcal{A}}_0$ *lies in the commutator subgroup* if one (and hence any) representative w of $[w]$ lies in the commutator subgroup of $\mathbb{F}_2$.

Definition 5.1.6. Define the map $\bar{\alpha} \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ as follows: Let $[w] \in \bar{\mathcal{A}}_0$. If $[w] = e$ set $\bar{\alpha}([w]) = e$. Else choose a representative $w \in \mathcal{A}$ of $[w]$ that starts with a power of $\mathtt{a}$ and, as w has even length, ends in a power of $\mathtt{b}$. Suppose that w starts with the letter $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and write $w = \mathtt{x}w'$ for $w' \in \mathcal{A}$ such that $\mathtt{x}w'$ is reduced. Then define $\bar{\alpha} \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ via

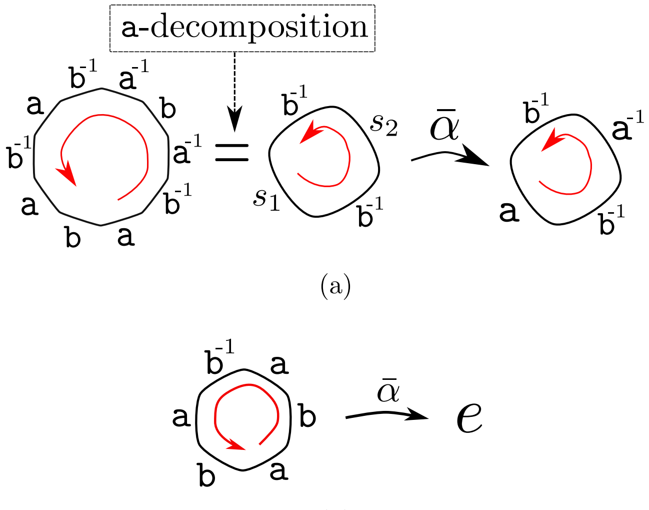
$$\bar{\alpha} \colon [w] \mapsto [\alpha(\mathtt{x}w'\mathtt{x})\mathtt{x}^{-1}] \in \bar{\mathcal{A}}_0.$$

Define $\bar{\beta} \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ analogously: For every element $[w] \in \bar{\mathcal{A}}_0$ choose a representative $w \in \mathcal{A}$ which starts with the letter $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$ and write $w = \mathtt{y}w'$. Then define $\bar{\beta} \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ via

$$\bar{\beta} \colon [w] \mapsto [\beta(\mathtt{y}w'\mathtt{y})\mathtt{y}^{-1}] \in \bar{\mathcal{A}}_0.$$

To see that $\bar{\alpha}, \bar{\beta} \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ are well-defined, suppose that $w_1, w_2 \in \mathcal{A}$ are both even alternating words which start in a power of $\mathtt{a}$ and both represent the same element $[w_1] = [w_2] \in \bar{\mathcal{A}}_0$. Let $\mathtt{x}_1, \mathtt{x}_2 \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ be the first letters of w_1 and w_2. Then there are elements $v_1, v_2 \in \mathcal{A}$ such that $w_1 = \mathtt{x}_1 v_1 \mathtt{x}_2 v_2$ as a reduced word and $w_2 = \mathtt{x}_2 v_2 \mathtt{x}_1 v_1$. Then, by (3) of Proposition 5.1.4,

$$\alpha(w_1\mathtt{x}_1)\mathtt{x}_1^{-1} = \alpha(\mathtt{x}_1 v_1 \mathtt{x}_2 v_2 \mathtt{x}_1)\mathtt{x}_1^{-1} = \alpha(\mathtt{x}_1 v_1 \mathtt{x}_2)\mathtt{x}_2^{-1}\alpha(\mathtt{x}_2 v_2 \mathtt{x}_1)\mathtt{x}_1^{-1}$$
$$\alpha(w_2\mathtt{x}_2)\mathtt{x}_2^{-1} = \alpha(\mathtt{x}_2 v_2 \mathtt{x}_1 v_1 \mathtt{x}_2)\mathtt{x}_2^{-1} = \alpha(\mathtt{x}_2 v_1 \mathtt{x}_1)\mathtt{x}_1^{-1}\alpha(\mathtt{x}_1 v_1 \mathtt{x}_2)\mathtt{x}_2^{-1}$$



(a) (b)

Figure 5.1: Visualizing $\bar{\alpha}$: Conjugacy classes $[w]$ correspond to cyclic labellings of a circle. One may define an $\mathtt{a}$-decomposition and $\bar{\alpha}$ on such labels except when $[w]$ does not contain $\mathtt{a}$ or $\mathtt{a}^{-1}$ as a subword. See Example 5.1.7

which are conjugate in $\mathbb{F}_2$ and so $[\alpha(w_1\mathtt{x}_1)\mathtt{x}_1^{-1}] = [\alpha(w_2\mathtt{x}_2)\mathtt{x}_2^{-1}]$. This shows that $\bar{\alpha}$ is well defined and analogously that $\bar{\beta}$ is well defined.

The definition of $\bar{\alpha}$ given above is useful for performing calculations. However, there is a more geometric way to think about $\bar{\alpha}$ and $\bar{\beta}$ analogous to the definition of α and β. A common way to depict conjugacy classes in the free group is via labels on a circle: Let $w = \mathtt{z}_1 \cdots \mathtt{z}_n \in \mathbb{F}_2$ be a cyclically reduced word in the letters $\mathtt{z}_i$. Then w labels a circle by cyclically labelling the sides of the circle counterclockwise by $\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n$ so that $\mathtt{z}_n$ is next to $\mathtt{z}_1$ on the circle. Two cyclically reduced words $w \in \mathbb{F}_2$ then yield the same labelling up to rotation if and only if they define the same conjugacy class.

Let $[w] \in \bar{\mathcal{A}}_0$ be a conjugacy class of a word $w \in \mathcal{A}$ of even length that contains both at least one $\mathtt{a}$ and one $\mathtt{a}^{-1}$ as a subword. We may similarly define an $\mathtt{a}$-decomposition of such a cyclic labelling. One may show that in this geometric model the maps $\bar{\alpha}$ (resp. $\bar{\beta}$) can then be defined just like for α and β by replacing the words in $\mathcal{S}_{\mathtt{a}}^+$ by $\mathtt{a}$ and the words in $\mathcal{S}_{\mathtt{a}}^-$ by $\mathtt{a}^-$. If $[w] \in \bar{\mathcal{A}}_0$ does not contain both $\mathtt{a}$ and $\mathtt{a}^{-1}$ as subwords then $\bar{\alpha}([w]) = e$ in both cases. Consider the following example:

Example 5.1.7. Let $w = \mathtt{ab^{-1}a^{-1}ba^{-1}b^{-1}ab^{-1}ab} \in \mathcal{A}$. Its conjugacy class is depicted in Figure 5.1. We observe that w starts with $\mathtt{a}$ and set $w' = \mathtt{b^{-1}a^{-1}ba^{-1}b^{-1}ab^{-1}ab}$ so that $w = \mathtt{a}w'$. By Definition 5.1.6, $\bar{\alpha}([w]) = [\alpha(\mathtt{a}w'\mathtt{a})\mathtt{a}^{-1}] = [(\mathtt{ab^{-1}a^{-1}b^{-1}a})\,\mathtt{a}^{-1}] = [\mathtt{ab^{-1}a^{-1}b^{-1}}]$. However, we could have also done an $\mathtt{a}$-decomposition of the elements on a circle as pictured in Figure 5.1 (A) with $s_1 = \mathtt{ab^{-1}aba} \in \mathcal{S}_{\mathtt{a}}^+$ and $s_2 = \mathtt{a^{-1}ba^{-1}} \in \mathcal{S}_{\mathtt{a}}^-$ and obtained the same result.

Similarly, let $w = \mathtt{abab^{-1}ab}$. It's conjugacy class is represented by a cyclic labelling of a circle in Figure 5.1 (B). The first letter of w is $\mathtt{a}$. Set $w' = \mathtt{bab^{-1}ab}$ so that $w = \mathtt{a}w'$. The $\mathtt{a}$-decomposition of $\mathtt{a}w'\mathtt{a} = s_1 \in \mathcal{S}_{\mathtt{a}}^+$. Hence $\bar{\alpha}([w]) = [\alpha(\mathtt{a}w'\mathtt{a})\mathtt{a}^{-1}] = [(\mathtt{a})\,\mathtt{a}^{-1}] = [e] \in \bar{\mathcal{A}}_0$.

Proposition 5.1.8. *Let $\bar{\alpha}, \bar{\beta} \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ be defined as above and let $[w] \in \bar{\mathcal{A}}_0$. Then $|\bar{\alpha}([w])| \leq |[w]|$ with equality if and only if $\bar{\alpha}([w]) = [w]$. The analogous statement holds for $\bar{\beta}$. If $[w]$ is a non-trivial class in the commutator subgroup of $\mathbb{F}_2$ then $\bar{\alpha}([w])$ and $\bar{\beta}([w])$ are non-trivial. If $\bar{\alpha}([w]) = [w] = \bar{\beta}([w])$ then $[w]$ may be represented by $w = [\mathtt{a}, \mathtt{b}]^n$ for $n \in \mathbb{Z}$.*

Proof. To see that $\bar{\alpha}$, $\bar{\beta}$ decrease length unless they fix classes is the same argument as in the proof of Proposition 5.1.4. If $[w]$ is a non-trivial class in the commutator subgroup of $\mathbb{F}_2$ then there is a reduced representative w such that $w = \mathtt{a}v_1\mathtt{a}^{-1}v_2$ for some appropriate $v_1, v_2 \in \mathcal{A}$ and we see that $\bar{\alpha}([w])$ is non-trivial as it also contains the subletters $\mathtt{a}$ and $\mathtt{a}^{-1}$. If $w \in \mathcal{A}$ is a representative such that $\bar{\alpha}$ fixes $[w]$ then w has to be of the form $w = \prod_{i=1}^k \mathtt{ay}_i\mathtt{a}^{-1}\mathtt{y}'_i$ for some $\mathtt{y}_i, \mathtt{y}'_i \in \{\mathtt{b}, \mathtt{b}^{-1}\}$, $k \geq 1$ and similarly, if $\bar{\beta}$ fixes a class then the a representative has to be of the form $w = \prod_{i=1}^k \mathtt{x}_i\mathtt{bx}'_i\mathtt{b}^{-1}$ for some $\mathtt{x}_i, \mathtt{x}'_i \in \{\mathtt{a}, \mathtt{a}^{-1}\}$, $k \geq 1$. Comparing both yields the statement. □

Proposition 5.1.9. *Assume that $w \in \mathcal{A}$ is non-empty, has even length and that $c_1, c_2 \in \mathcal{A}$ are words such that $c_1wc_2 \in \mathcal{A}$ is again an alternating word. Then there are words $d_1, d_2, w' \in \mathcal{A}$ such that $\alpha(c_1w^nc_2) = d_1w'^{n-1}d_2 \in \mathcal{A}$ for all $n \geq 1$ as reduced words where w' has even length and $[w'] = \bar{\alpha}([w]) \in \bar{\mathcal{A}}_0$. If w lies in the commutator subgroup then w' is non-empty. The analogous statement holds for β.*

Proof. If $w \in \mathcal{A}$ does not contain both a positive and a negative power of $\mathtt{a}$, the statement follows by an easy calculation. Note that this is the case if and only if $\bar{\alpha}([w]) = [e]$. Otherwise w contains at least one sub-letter $\mathtt{a}$ and one sub-letter $\mathtt{a}^{-1}$. This is the case if w lies in the commutator subgroup. Suppose without loss of generality that $w = v_1 \mathtt{a} v_2 \mathtt{a}^{-1} v_3$ as a reduced word for some $v_1, v_2, v_3 \in \mathcal{A}$. By multiple applications of Proposition 5.1.4, we see that

$$\begin{aligned}
\alpha(c_1 w^n c_2) &= \alpha(c_2 \left(v_1 \mathtt{a} v_2 \mathtt{a}^{-1} v_3\right)^n c_2) \\
&= \alpha(c_1 v_1 \mathtt{a})\mathtt{a}^{-1} \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 (v_1 \mathtt{a} v_2 \mathtt{a}^{-1} v_3)^{n-1} c_2) \\
&= \alpha(c_1 v_1 \mathtt{a})\mathtt{a}^{-1} \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})\mathtt{a}^{-1} \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 (v_1 \mathtt{a} v_2 \mathtt{a}^{-1} v_3)^{n-2} c_2) \\
&= \alpha(c_1 v_1 \mathtt{a})\mathtt{a}^{-1} \left(\alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})\mathtt{a}^{-1}\right)^2 \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 (v_1 \mathtt{a} v_2 \mathtt{a}^{-1} v_3)^{n-3} c_2) \\
&= \cdots \\
&= \alpha(c_1 v_1 \mathtt{a})\mathtt{a}^{-1} (\alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})\mathtt{a}^{-1})^{n-1} \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 c_2)
\end{aligned}$$

as non-reduced elements in the free group. Then we define d_1, d_2 and w' to be the reduced representative of

$$\alpha(c_1 v_1 \mathtt{a})\mathtt{a}^{-1}, \ \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 c_2) \text{ and } \alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})\mathtt{a}^{-1}$$

respectively. Moreover, $\alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})$ is a reduced alternating word which starts and ends in $\mathtt{a}$ and contains the $\mathtt{a}^{-1}$ as a sub-letter. If follows that w', the reduced representative of $\alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})\mathtt{a}^{-1}$, starts with $\mathtt{a}$, contains $\mathtt{a}^{-1}$ and ends with a power of $\mathtt{b}$, so w' is non-empty. Further observe that $\bar{\alpha}([\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1])$ is represented by $\alpha(\mathtt{a} v_2 \mathtt{a}^{-1} v_3 v_1 \mathtt{a})\mathtt{a}^{-1}$ and hence $[w'] = \bar{\alpha}(w)$. □

5.1.2 Letter-Thin Triples, α and β

In order to streamline proofs later and ease notation we define an equivalence relation on triples (x_1, x_2, x_3). We think of such a triple as the sides of a (thin) triangle. We stress that the x_i are not actually the side of triangles in some metric space; see Figure 5.2. Here, we study a special type of triples, namely *letter-thin triples* in Definition 5.1.12.

Definition 5.1.10. Let (x_1, x_2, x_3) be a triple of elements in $\mathbb{F}_2$ and let $\phi \colon \mathbb{F}_2 \to \mathbb{F}_2$ be a set-theoretic function. We will understand by $\phi(x_1, x_2, x_3)$ the triple $(\phi(x_1), \phi(x_2), \phi(x_3))$. We define $\sim$ to be the equivalence relation on triples generated by

(i) $(x_1, x_2, x_3) \sim (x_2, x_3, x_1)$

(ii) $(x_1, x_2, x_3) \sim (x_3^{-1}, x_2^{-1}, x_1^{-1})$

(iii) $(x_1, x_2, x_3) \sim \phi_{\mathtt{a}}(x_1, x_2, x_3)$, where $\phi_{\mathtt{a}} \colon \mathbb{F}_2 \to \mathbb{F}_2$ is the automorphism defined via $\mathtt{a} \mapsto \mathtt{a}^{-1}$ and $\mathtt{b} \mapsto \mathtt{b}$.

(iv) $(x_1, x_2, x_3) \sim \phi_{\mathtt{b}}(x_1, x_2, x_3)$, where $\phi_{\mathtt{b}} \colon \mathbb{F}_2 \to \mathbb{F}_2$ is the automorphism defined via $\mathtt{a} \mapsto \mathtt{a}$ and $\mathtt{b} \mapsto \mathtt{b}^{-1}$.

for all $x_1, x_2, x_3 \in \mathbb{F}_2$ and say that (x_1, x_2, x_3) is *equivalent* to (y_1, y_2, y_3) if $(x_1, x_2, x_3) \sim (y_1, y_2, y_3)$ under this relation.

Imagining (x_1, x_2, x_3) as labelling the sides of a triangle, two triples are equivalent if they may be obtained from each other by a sequence of rotations (i), flips (ii) or by changing the signs of its labels (iii) & (iv).

Proposition 5.1.11. *Let $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{F}_2$ such that $(x_1, x_2, x_3) \sim (y_1, y_2, y_3)$. Then if $x_1, x_2, x_3 \in \mathcal{A}$ also $y_1, y_2, y_3 \in \mathcal{A}$. Moreover, in this case $\alpha(x_1, x_2, x_3) \sim \alpha(y_1, y_2, y_3)$ and $\beta(x_1, x_2, x_3) \sim \beta(y_1, y_2, y_3)$.*

Proof. The first part is clear from the definitions. Note that α commutes both with "rotating the side" (i) and taking inverses (ii) as α satisfies that $\alpha(w^{-1}) = \alpha(w)^{-1}$ for $w \in \mathcal{A}$.

Let $w = \mathtt{y}_0 s_1 \mathtt{y}_1 \cdots \mathtt{y}_{k-1} s_k \mathtt{y}_k$ be the $\mathtt{a}$-decomposition of w (see Definition 5.1.2), where $\mathtt{y}_i \in \{\mathtt{b}, \mathtt{b}^{-1}\}$ and $s_i \in \mathcal{S}_{\mathtt{a}}^+ \cup \mathcal{S}_{\mathtt{a}}^-$ alternates between $\mathcal{S}_{\mathtt{a}}^+$ and $\mathcal{S}_{\mathtt{a}}^-$. Then

$$\phi_{\mathtt{a}}(w) = \mathtt{y}_0 \phi_{\mathtt{a}}(s_1) \mathtt{y}_1 \cdots \mathtt{y}_{k-1} \phi_{\mathtt{a}}(s_k) \mathtt{y}_k$$

where $\phi(s_i) \in \mathcal{S}_{\mathtt{a}}^+$ if and only if $s_i \in \mathcal{S}_{\mathtt{a}}^-$ and $\phi(s_i) \in \mathcal{S}_{\mathtt{a}}^-$ if and only if $s_i \in \mathcal{S}_{\mathtt{a}}^+$. So $\alpha(\phi_{\mathtt{a}}(w)) = \phi_{\mathtt{a}}(\alpha(w))$ and hence $\alpha \circ \phi_{\mathtt{a}}(x_1, x_2, x_3)$ is equivalent to $\alpha(x_1, x_2, x_3)$. Similarly, $\phi_{\mathtt{b}}(w) = \phi_{\mathtt{b}}(\mathtt{y}_0)\phi_{\mathtt{b}}(s_1)\phi_{\mathtt{b}}(\mathtt{y}_1) \cdots \phi_{\mathtt{b}}(\mathtt{y}_{k-1})\phi_{\mathtt{b}}(s_k)\phi_{\mathtt{b}}(\mathtt{y}_k)$ where both $\phi_{\mathtt{b}}(s_i)$ and s_i lie in the same set $\mathcal{S}_{\mathtt{a}}^+$ or $\mathcal{S}_{\mathtt{a}}^-$. We see that once more, $\alpha(\phi_{\mathtt{b}}(w)) = \phi_{\mathtt{b}}(\alpha(w))$ and hence also $\alpha \circ \phi_{\mathtt{b}}(x_1, x_2, x_3)$ is equivalent to $\alpha(x_1, x_2, x_3)$. Analogously, we see the statement for β. □

For a visualisation of the following definition we refer the reader to Figure 5.2.

Definition 5.1.12. Let $x_1, x_2, x_3 \in \mathcal{A}$ be *alternating* elements. The triple (x_1, x_2, x_3) is called *letter-thin triple* in one of the following cases:

[T1] There are (possibly trivial) elements $c_1, c_2, c_3 \in \mathcal{A}$ such that

[T1$\mathtt{a}$] $(x_1, x_2, x_3) \sim (c_1^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}\mathtt{a}c_3, c_3^{-1}\mathtt{a}^{-1}c_1)$ or

[T1$\mathtt{b}$] $(x_1, x_2, x_3) \sim (c_1^{-1}\mathtt{ba}c_2, c_2^{-1}\mathtt{a}^{-1}\mathtt{b}c_3, c_3^{-1}\mathtt{b}^{-1}c_1)$

where all words are required to be reduced.

[T2] There are (possibly trivial) elements $c_1, c_2 \in \mathcal{A}$ such that

[T2$\mathtt{a}$] $(x_1, x_2, x_3) \sim (c_1^{-1}\mathtt{b}^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}, \mathtt{b}c_1)$ or

[T2$\mathtt{b}$] $(x_1, x_2, x_3) \sim (c_1^{-1}\mathtt{a}^{-1}\mathtt{ba}c_2, c_2^{-1}\mathtt{a}^{-1}, \mathtt{a}c_1)$

where all words are required to be reduced.

In all cases, $\sim$ denotes the equivalence of triples of Definition 5.1.10. We say that a letter-thin triple (x_1, x_2, x_3) is *of type [T1$\mathtt{a}$], [T1$\mathtt{b}$], [T2$\mathtt{a}$]* or *[T2$\mathtt{b}$]* if it is equivalent to the corresponding triple above.

Note for example in the representatives of [T1$\mathtt{a}$] above, necessarily c_1, c_3 are either empty or their first letter is a power of $\mathtt{b}$. Similarly, c_2 is either empty or its first letter is a power of $\mathtt{a}$, else the x_i would not be alternating.

Note that for any letter-thin triple (x_1, x_2, x_3) of type [T1$\mathtt{a}$] we may always find elements $d_1, d_2, d_3 \in \mathcal{A}$ with first letter a power of $\mathtt{b}$ such that

$$(x_1, x_2, x_3) = (d_1^{-1}\mathtt{x}_1 d_2, d_2^{-1}\mathtt{x}_2 d_3, d_3^{-1}\mathtt{x}_3 d_1) \tag{5.1}$$

where $\mathtt{x}_i \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ are such that *not all of $\mathtt{x}_1$, $\mathtt{x}_2$ and $\mathtt{x}_3$ are equal* i.e. have the same parity. As we consider the triples only up to equivalence one may wonder if we can assume that any triple as in Equation (5.1) such that not all of d_i are empty is letter-thin of type [T1$\mathtt{a}$]. However, this is not the case: As $\mathtt{x}_1$, $\mathtt{x}_2$, $\mathtt{x}_3$ do not all have the same parity, there is exactly one i such that $\mathtt{x}_i = \mathtt{x}_{i+1}$ where indices are considered $\mod 3$. Then one may see that (x_1, x_2, x_3) is of type [T1$\mathtt{a}$] *if and only if* d_{i+1} is non-trivial. For example, $(d_1^{-1}\mathtt{a}, \mathtt{a}d_3, d_3^{-1}\mathtt{a}^{-1}d_1)$ is *not* letter-thin for any $d_1, d_3 \in \mathcal{A}$ empty or starting with a power of $\mathtt{b}$.

Example 5.1.13. $(\mathtt{a}, \mathtt{a}, \mathtt{a}^{-1})$ is not letter-thin and by the previous discussion also the triple $(\mathtt{b}^{-1}\mathtt{a}^{-1}, \mathtt{a}^{-1}\mathtt{b}, \mathtt{b}^{-1}\mathtt{ab})$ is not letter-thin. However, $(\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{b}, \mathtt{b}^{-1}\mathtt{a}^{-1}, \mathtt{ab})$ *is* letter-thin. To see this, note that

$$\begin{aligned}(\mathtt{b}^{-1}\mathtt{a}^{-1}\mathtt{b}, \mathtt{b}^{-1}\mathtt{a}^{-1}, \mathtt{ab}) &\overset{(iii)}{\sim} (\mathtt{b}^{-1}\mathtt{ab}, \mathtt{b}^{-1}\mathtt{a}, \mathtt{a}^{-1}\mathtt{b}) \\ &= (c_1^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}\mathtt{a}c_3, c_3^{-1}\mathtt{a}^{-1}c_1)\end{aligned}$$

for $c_1 = \mathtt{b}$, $c_2 = e$ and $c_3 = e$ and where $\overset{(iii)}{\sim}$ denotes the equivalence (iii) of the definition of '$\sim$'; see Definition 5.1.10.

Note that by definition, if (x_1, x_2, x_3) is letter-thin then *all* x_1, x_2, x_3 *are alternating words.*

See Figure 5.2 for the explanation of the name *letter-thin triple*: First consider elements $g, h \in \mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle$. The triple $(g, h, (gh)^{-1})$ corresponds to sides of a geodesic triangle in the Cayley graph $\mathrm{Cay}(\mathbb{F}_2, \{\mathtt{a}, \mathtt{b}\})$ with endpoints e, g, gh. Note further that there are words $c_1, c_2, c_3 \in \mathbb{F}_2$ such that $g = c_1^{-1}c_2$, $h = c_2^{-1}c_3$, $(gh)^{-1} = c_3^{-1}c_1$ and all these expressions are freely reduced. A *letter-thin* triple (x_1, x_2, x_3) is such that each x_i is in addition alternating and corresponds *almost* to the sides of a geodesic triangle in a Cayley graph, apart from one letter $r \in \{\mathtt{a}, \mathtt{b}\}$ in the "middle" of the triangle. Figure 5.2 (B) corresponds to case [T1] of Definition 5.1.12, Figure 5.2 (C) corresponds to case [T2] of Definition 5.1.12. These letter-thin triples (x_1, x_2, x_3) do *not* label sides of triangles in a Cayley graph or any other metric space.

Observe that (x_1, x_2, x_3) is letter-thin if and only if $\psi(x_1, x_2, x_3)$ is letter-thin for ψ defined as in Proposition 5.1.4 (2) i.e. ψ is the automorphism $\psi\colon \mathbb{F}_2 \to \mathbb{F}_2$ defined via $\psi\colon \mathtt{a} \mapsto \mathtt{b}$ and $\psi\colon \mathtt{b} \mapsto \mathtt{a}$.

The maps α and β respect letter-thin triples:

Lemma 5.1.14. *If (x_1, x_2, x_3) is letter-thin. Then both $\alpha(x_1, x_2, x_3)$ and $\beta(x_1, x_2, x_3)$ are letter-thin.*

Proof. We will proceed as follows: Let (x_1, x_2, x_3) be a letter-thin triple. By Proposition 5.1.11 it is enough to check that $\alpha(x_1, x_2, x_3)$ is letter-thin for one representative of the equivalence class. Hence it suffices to check that $\alpha(x_1, x_2, x_3)$ is letter thin for

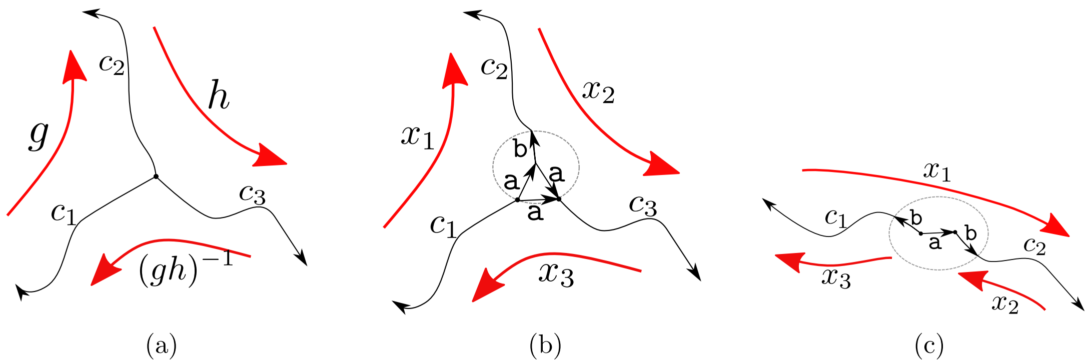


Figure 5.2: Different "triangles": (A) arises as a generic thin triangle in the Cayley graph $\mathrm{Cay}(\mathbb{F}_2, \{\mathtt{a}, \mathtt{b}\})$ of the free group with standard generating set. Figures (B) and (C) correspond to letter-thin triples [T1a], [T2a]. The grey dotted circles indicate the part of the letter-thin triples which can not be empty. These letter-thin triples do *not* generally live in a Cayley graph or any well-known metric space.

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1. Type [T1a]: $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}\mathtt{a}c_3, c_3^{-1}\mathtt{a}^{-1}c_1)$
2. Type [T1b]: $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{ba}c_2, c_2^{-1}\mathtt{a}^{-1}\mathtt{b}c_3, c_3^{-1}\mathtt{b}^{-1}c_1)$
3. Type [T2a]: $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{b}^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}, \mathtt{b}c_1)$
4. Type [T2b]: $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{a}^{-1}\mathtt{ba}c_2, c_2^{-1}\mathtt{a}^{-1}, \mathtt{a}c_1)$

By symmetry, this will show the analogous statement for β.

Proposition 5.1.4, (4) allows us to compute α piecewise i.e. after each occurrence of a letter $\mathtt{a}$ or $\mathtt{a}^{-1}$ in a reduced word. For any reduced word $c \in \mathcal{A}$ starting with a power of $\mathtt{b}$ or being empty, we will write c_+ for the reduced word represented by $\mathtt{a}^{-1}\alpha(\mathtt{a}c)$, which itself is not reduced since $\alpha(\mathtt{a}c)$ starts with an $\mathtt{a}$. Similarly, we will write c_- for the reduced word represented by $\mathtt{a}\alpha(\mathtt{a}^{-1}c)$. Note that c_+ and c_- are either empty or their first letter is a power of $\mathtt{b}$, as $\alpha(\mathtt{a}^{\pm}c)$ is alternating. If c is a word which already has a subscript, say c_i, then we will write $c_{i,+}$ and $c_{i,-}$, respectively. We consider each of the above cases independently. For letter-thin triples (x_1, x_2, x_3) of type [T1a] we compute $\alpha(x_1, x_2, x_3)$ and we will state exactly which equivalences (i), (ii), (iii) and (iv) of Definition 5.1.10 are needed to obtain one of the representatives for [T1a], [T1b], [T2a] and [T2b] of letter-thin triples as in Definition 5.1.12. For letter-thin triples (x_1, x_2, x_3) of type [T1$\mathtt{b}$], [T2$\mathtt{a}$] and [T2$\mathtt{b}$] we will just state the type of $\alpha(x_1, x_2, x_3)$ without explicitly giving the equivalence.

1. Type [T1$\mathtt{a}$]: Suppose $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}\mathtt{a}c_3, c_3^{-1}\mathtt{a}^{-1}c_1)$. As (x_1, x_2, x_3) are alternating c_2 is either empty or starts with a positive or a negative power of $\mathtt{a}$. We consider these cases separately:

   - c_2 is empty. In this case we compute using Proposition 5.1.4,

     $$\alpha(c_1^{-1}\mathtt{ab}) = \alpha(c_1^{-1}\mathtt{a})\mathtt{a}^{-1}\alpha(\mathtt{ab}) = \alpha(\mathtt{a}^{-1}c_1)^{-1}\mathtt{b} = (\mathtt{a}^{-1}c_{1,-})^{-1}\mathtt{b} = (c_{1,-})^{-1}\mathtt{ab}$$
     $$\alpha(\mathtt{b}^{-1}\mathtt{a}c_3) = \alpha(\mathtt{b}^{-1}\mathtt{a})\mathtt{a}^{-1}\alpha(\mathtt{a}c_3) = \mathtt{b}^{-1}\mathtt{a}c_{3,+}$$
     $$\alpha(c_3^{-1}\mathtt{a}^{-1}c_1) = \alpha(c_3^{-1}\mathtt{a}^{-1})\mathtt{a}\alpha(\mathtt{a}^{-1}c_1) = \alpha(\mathtt{a}c_3)^{-1}c_{1,-} = (\mathtt{a}c_{3,+})^{-1}c_{1,-} = (c_{3,+})^{-1}\mathtt{a}^{-1}c_{1,-}$$

     and hence

     $$\alpha(x_1, x_2, x_3) = ((c_{1,-})^{-1}\mathtt{ab}, \mathtt{b}^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}c_{1,-})$$

     which is of type [T1$\mathtt{a}$]. Indeed, for $c_1' = c_{1,-}$, $c_2' = e$ and $c_3' = c_{3,+}$ we see that

     $$\alpha(x_1, x_2, x_3) = (c_1'^{-1}\mathtt{ab}c_2', c_2'^{-1}\mathtt{b}^{-1}\mathtt{a}c_3', c_3'^{-1}\mathtt{a}^{-1}c_1').$$

     and hence $\alpha(x_1, x_2, x_3)$ is of type [T1$\mathtt{a}$].

   - $c_2 = \mathtt{a}d_2$ where, $d_2 \in \mathcal{A}$.

     $$\alpha(x_1, x_2, x_3) = ((c_{1,-})^{-1}\mathtt{a}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}c_{1,-})$$

     which is of type [T2$\mathtt{b}$] if $c_{1,-}$ is trivial and of type [T1$\mathtt{b}$] else. To see this we distinguish between three different cases:

     – $c_{1,-}$ is trivial: Then

     $$\begin{aligned}\alpha(x_1, x_2, x_3) &= (\mathtt{a}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}) \\ &\overset{(i)}{\sim} ((d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}, \mathtt{a}d_{2,+}) \\ &\overset{(iv)}{\sim} (\phi_b(d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}c_{3,+}, \phi_b(c_{3,+})^{-1}\mathtt{a}^{-1}, \mathtt{a}\phi_b(d_{2,+})) \\ &= (c_1'^{-1}\mathtt{a}^{-1}\mathtt{ba}c_2', c_2'^{-1}\mathtt{a}^{-1}, \mathtt{a}c_1')\end{aligned}$$

for $c_1' = \phi_b(d_{2,+})^{-1}$ and $c_2' = c_{3,+}$ and hence of type [T2$\mathtt{b}$]. Here $\sim$ denotes the equivalences on triples defined in Definition 5.1.10 with the corresponding numbering $(i) - (iv)$.

- $c_{1,-}$ is non-trivial and starts with first letter $\mathtt{b}$. Then define d_1 via $c_{1,-} = \mathtt{b}d_1$. Hence $\alpha(x_1, x_2, x_3)$ equals:

$$\begin{aligned}&(d_1^{-1}\mathtt{b}^{-1}\mathtt{a}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}\mathtt{b}d_1)\\ \overset{(iv)}{\sim}&(\phi_b(d_1)^{-1}\mathtt{ba}\phi_b(d_{2,+}), \phi_b(d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}\phi_b(c_{3,+}), \phi_b(c_{3,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\phi_b(d_1))\\ =&(c_1'^{-1}\mathtt{ba}c_2', c_2'^{-1}\mathtt{a}^{-1}\mathtt{b}c_3', c_3'^{-1}\mathtt{b}^{-1}c_1')\end{aligned}$$

  for $c_1' = \phi_b(d_1)$, $c_2' = \phi_b(d_{2,+})$, $c_3' = \mathtt{a}\phi_b(c_{3,+})$ and hence is of type [T1$\mathtt{b}$].

- $c_{1,-}$ is non-trivial and starts with first letter $\mathtt{b}^{-1}$. Then define d_1 via $c_{1,-} = \mathtt{b}^{-1}d_1$. Hence $\alpha(x_1, x_2, x_3)$ equals:

$$\begin{aligned}&(d_1^{-1}\mathtt{ba}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}d_1)\\ \overset{(ii)}{\sim}&d_1^{-1}\mathtt{ba}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}d_1)\\ =&(c_1'^{-1}\mathtt{ba}c_2', c_2'^{-1}\mathtt{a}^{-1}\mathtt{b}c_3', c_3'^{-1}\mathtt{b}^{-1}c_1')\end{aligned}$$

  for $c_1' = d_1$, $c_2' = c_{3,+}$, $c_3' = \mathtt{a}d_{2,+}$ and hence of type [T1$\mathtt{b}$].

- $c_2 = \mathtt{a}^{-1}d_2$ where $d_2 \in \mathcal{A}$.

$$\alpha(x_1, x_2, x_3) = ((c_{1,-})^{-1}\mathtt{aba}^{-1}d_{2,-}, (d_{2,-})^{-1}\mathtt{a}c_{3,+}, (c_{3,+})^{-1}\mathtt{a}^{-1}c_{1,-})$$

  which is of type [T1$\mathtt{b}$] if $c_{3,+}$ is non-trivial and of type [T2$\mathtt{b}$], else. This can be seen analogously to the previous case.

2. Type [T1$\mathtt{b}$]: Suppose $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{ba}c_2, c_2^{-1}\mathtt{a}^{-1}\mathtt{b}c_3, c_3^{-1}\mathtt{b}^{-1}c_1)$. Up to equivalence, there are the following sub-cases:

   - Both of c_1, c_3 are empty. Then

$$\alpha(x_1, x_2, x_3) = (\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}, \mathtt{b}^{-1})$$

     which is of type [T1$\mathtt{b}$]

   - c_1 is not empty, c_3 is empty. Then either

- $c_1 = \mathtt{a}d_1$. In this case

$$\alpha(x_1, x_2, x_3) = ((d_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}, \mathtt{b}^{-1}\mathtt{a}d_{1,+})$$

which is of type [T1$\mathtt{b}$]

- $c_1 = \mathtt{a}^{-1}d_1$. In this case

$$\alpha(x_1, x_2, x_3) = ((d_{1,-})^{-1}\mathtt{a}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}, \mathtt{b}^{-1}\mathtt{a}^{-1}d_{1,+})$$

which is of type [T1$\mathtt{a}$].

- c_1 is empty and c_3 is not. Then either
  - $c_3 = \mathtt{a}d_3$, in which case

$$\alpha(x_1, x_2, x_3) = (\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}d_{3,+}, (d_{3,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1})$$

which is of type [T1$\mathtt{b}$].

  - $c_3 = \mathtt{a}^{-1}d_3$, in which case

$$\alpha(x_1, x_2, x_3) = (\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}d_{3,-}, (d_{3,-})^{-1}\mathtt{ab}^{-1})$$

which is of type [T1$\mathtt{a}$].

- Both of c_1, c_3 are non-empty. Then either
  - $c_1 = \mathtt{a}d_1$, $c_3 = \mathtt{a}d_3$. In this case

$$\alpha(x_1, x_2, x_3) = ((d_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}d_{3,+}, (d_{3,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}d_{1,+})$$

which is of type [T1$\mathtt{b}$].

  - $c_1 = \mathtt{a}d_1$, $c_3 = \mathtt{a}^{-1}d_3$. In this case

$$\alpha(x_1, x_2, x_3) = ((d_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}d_{3,-}, (d_{3,-})^{-1}\mathtt{a}d_{1,+})$$

which is of type [T1$\mathtt{b}$] if $d_{3,-}$ is non-trivial, and of type [T2$\mathtt{b}$], else.

  - $c_1 = \mathtt{a}^{-1}d_1$, $c_3 = \mathtt{a}d_3$. In this case

$$\alpha(x_1, x_2, x_3) = ((d_{1,-})^{-1}\mathtt{a}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}d_{3,+}, (d_{3,+})^{-1}\mathtt{a}^{-1}d_{1,-})$$

which is of type [T1$\mathtt{b}$] if $d_{1,-}$ is non-trivial and of type [T2$\mathtt{b}$], else.

– $c_1 = \mathtt{a}^{-1} d_1$, $c_3 = \mathtt{a}^{-1} d_3$. In this case

$$\alpha(x_1, x_2, x_3) = ((d_{1,-})^{-1}\mathtt{a}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}\, d_{3,-}, (d_{3,-})^{-1}\mathtt{ab}^{-1}\mathtt{a}^{-1}d_{1,-})$$

which is of type [T1$\mathtt{b}$] if $c_{2,+}$ is non-trivial and of type [T2$\mathtt{b}$], else.

3. Type [T2$\mathtt{a}$]: Suppose $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{b}^{-1}\mathtt{ab}c_2, c_2^{-1}\mathtt{b}^{-1}, \mathtt{b}c_1)$. We distinguish between the following cases

   - Both of c_1, c_2 are empty. Then

     $$\alpha(x_1, x_2, x_3) = (\mathtt{b}^{-1}\mathtt{ab}, \mathtt{b}^{-1}, \mathtt{b})$$

     which is of type [T2$\mathtt{a}$].

   - One of c_1, c_2 is empty. Up to equivalence and changing indices we may assume that c_2 is empty. Then either

     – $c_1 = \mathtt{a}d_1$ in which case

     $$\alpha(x_1, x_2, x_3) = ((d_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{ab}, \mathtt{b}^{-1}, \mathtt{ba}d_{1,+})$$

     which is of type [T2$\mathtt{a}$] or

     – $c_1 = \mathtt{a}^{-1}d_1$ in which case

     $$\alpha(x_1, x_2, x_3) = ((d_{1,-})^{-1}\mathtt{ab}, \mathtt{b}^{-1}, \mathtt{ba}^{-1}d_{1,-})$$

     which is of type [T1$\mathtt{b}$].

   - Both of c_1, c_2 are non-empty. Then either

     – $c_1 = \mathtt{a}d_1$, $c_2 = \mathtt{a}d_2$ in which case

     $$\alpha(x_1, x_2, x_3) = ((d_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{a}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}, \mathtt{ba}d_{1,+})$$

     which is of type [T1$\mathtt{b}$] or

     – $c_1 = \mathtt{a}d_1$, $c_2 = \mathtt{a}^{-1}d_2$ in which case

     $$\alpha(x_1, x_2, x_3) = ((d_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}\mathtt{aba}^{-1}d_{2,-}, (d_{2,-})^{-1}\mathtt{ab}^{-1}, \mathtt{ba}d_{1,+})$$

     which is of type [T2$\mathtt{a}$] or

– $c_1 = \mathtt{a}^{-1}d_1$, $c_2 = \mathtt{a}d_2$ in which case

$$\alpha(x_1, x_2, x_3) = ((d_{1,-})^{-1}\mathtt{a}d_{2,+}, (d_{2,+})^{-1}\mathtt{a}^{-1}\mathtt{b}^{-1}, \mathtt{ba}^{-1}d_{1,-})$$

which is of type [T1a] or

– $c_1 = \mathtt{a}^{-1}d_1$, $c_2 = \mathtt{a}^{-1}d_2$ in which case

$$\alpha(x_1, x_2, x_3) = ((d_{1,-})^{-1}\mathtt{aba}^{-1}d_{2,-}, (d_{2,-})^{-1}\mathtt{ab}^{-1}, \mathtt{ba}^{-1}d_{1,-})$$

which is of type [T1b].

4. Type [T2b]: Suppose $(x_1, x_2, x_3) = (c_1^{-1}\mathtt{a}^{-1}\mathtt{ba}c_2, c_2^{-1}\mathtt{a}^{-1}, \mathtt{a}c_1)$. We see that

$$\alpha(x_1, x_2, x_3) = ((c_{1,+})^{-1}\mathtt{a}^{-1}\mathtt{ba}c_{2,+}, (c_{2,+})^{-1}\mathtt{a}^{-1}, \mathtt{a}c_{1,+})$$

which is of type [T2b].

This concludes the proof of Lemma 5.1.14. □

5.1.3 Brooks Quasimorphisms, Homomorphisms and Letter-Thin Triples

For what follows we want to study how the Brooks quasimorphism $\eta_0 = \eta_{\mathtt{ab}} - \eta_{\mathtt{ba}}$ defined in Example 2.3.3 or certain homomorphisms behave on letter-thin triples. This will be done in Propositions 5.1.15 and 5.1.16, respectively.

Proposition 5.1.15. *Let $\eta_0 = \eta_{ab} - \eta_{ba} \colon \mathbb{F}_2 \to \mathbb{Z}$ be as above. Then*

$$|\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3)| = 1$$

for every letter-thin triple (x_1, x_2, x_3). In particular $\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3) \in \{-1, +1\}$.

Proof. First note that if $w = w_1 w_2 \in \mathbb{F}_2$ as a reduced word and if $\mathtt{z}_1$ is the last letter of w_1 and $\mathtt{z}_2$ is the first letter of w, then

$$\eta_0(w) = \eta_0(w_1) + \eta_0(\mathtt{z}_1\mathtt{z}_2) + \eta_0(w_2). \tag{5.2}$$

Let (x_1, x_2, x_3) be a triple. Note that the value

$$|\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3)|$$

is invariant under the equivalences (i) and (ii) of Definition 5.1.10. Up to these equivalences we see that any letter-thin triple (x_1, x_2, x_3) is equivalent via (i) and (ii) to the following:

- Type [T1a]: $(c_1^{-1}\mathtt{xy}c_2, c_2^{-1}\mathtt{y}^{-1}\mathtt{x}c_3, c_3^{-1}\mathtt{x}^{-1}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. If c_i is empty set $\mathtt{z}_i = e$. Else let $\mathtt{z}_i$ be the first letter of c_i. Then, by using successively Equation (5.2) we see that

  $$\begin{aligned} \eta_0(x_1) &= \eta_0(c_1^{-1}) + \eta_0(\mathtt{z}_1^{-1}\mathtt{x}) + \eta_0(\mathtt{xy}) + \eta_0(\mathtt{yz}_2) + \eta_0(c_2) \\ \eta_0(x_2) &= \eta_0(c_2^{-1}) + \eta_0(\mathtt{z}_2^{-1}\mathtt{y}^{-1}) + \eta_0(\mathtt{y}^{-1}\mathtt{x}) + \eta_0(\mathtt{xz}_3) + \eta_0(c_3) \\ \eta_0(x_3) &= \eta_0(c_3^{-1}) + \eta_0(\mathtt{z}_3^{-1}\mathtt{x}^{-1}) + \eta_0(\mathtt{x}^{-1}\mathtt{z}_1) + \eta_0(c_1) \end{aligned}$$

  Using that $\eta_0(c^{-1}) = -\eta_0(c)$ for any $c \in \mathbb{F}_2$ we see that

  $$|\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3)| = |\eta_0(\mathtt{xy}) + \eta_0(\mathtt{y}^{-1}\mathtt{x})|$$

  and hence we see that for any choice $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$, $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$

  $$|\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3)| = 1.$$

- Type [T1b]: $(c_1^{-1}\mathtt{yx}c_2, c_2^{-1}\mathtt{x}^{-1}\mathtt{y}c_3, c_3^{-1}\mathtt{y}^{-1}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. This case is analogous to the previous case.

- Type [T2a]: $(c_1^{-1}\mathtt{y}^{-1}\mathtt{xy}c_2, c_2^{-1}\mathtt{y}^{-1}, \mathtt{y}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. Again, if c_i is empty set $\mathtt{z}_i = e$. Else let $\mathtt{z}_i$ be the first letter of c_i. By successively using Equation (5.2) we see that

  $$\begin{aligned} \eta_0(x_1) &= \eta_0(c_1^{-1}) + \eta_0(\mathtt{z}_1^{-1}\mathtt{y}^{-1}) + \eta_0(\mathtt{y}^{-1}\mathtt{x}) + \eta_0(\mathtt{xy}) + \eta_0(\mathtt{yz}_2) + \eta_0(c_2) \\ \eta_0(x_2) &= \eta_0(c_2^{-1}) + \eta_0(\mathtt{z}_2^{-1}\mathtt{y}^{-1}) \\ \eta_0(x_3) &= \eta_0(\mathtt{yz}_1) + \eta_0(c_1) \end{aligned}$$

  and again we observe that

  $$|\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3)| = |\eta_0(\mathtt{y}^{-1}\mathtt{x}) + \eta_0(\mathtt{xy})| = 1$$

  for any choice of $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$, $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$.

- Type [T2b]: $(c_1^{-1}\mathtt{x}^{-1}\mathtt{y}\mathtt{x}\mathtt{b}c_2, c_2^{-1}\mathtt{x}^{-1}, \mathtt{x}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. This case is analogous to the previous case.

□

Recall that $\eta_{\mathtt{x}} \colon \mathbb{F}_2 \to \mathbb{Z}$ denotes the homomorphism which counts the letter $\mathtt{x}$.

Proposition 5.1.16. *Let $\eta = \eta_{\mathtt{x}} + \eta_{\mathtt{y}} \colon \mathbb{F}_2 \to \mathbb{Z}$ for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ or $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. Then*

$$|\eta(x_1) + \eta(x_2) + \eta(x_3)| = 1$$

for any (x_1, x_2, x_3) letter-thin. In particular $\eta(x_1) + \eta(x_2) + \eta(x_3) \in \{-1, +1\}$.

Proof. Let η be as in the proposition and suppose that (x_1, x_2, x_3) is letter-thin. Just like in the proof of the previous proposition we will consider the four different types of letter thin triples up to equivalences (i) and (ii) of Definition 5.1.10.

- Type [T1a]: $(c_1^{-1}\mathtt{x}\mathtt{y}c_2, c_2^{-1}\mathtt{y}^{-1}\mathtt{x}c_3, c_3^{-1}\mathtt{x}^{-1}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. We directly calculate, using that η is a homomorphism:

$$\begin{aligned}
\eta(x_1) &= \eta(c_1^{-1}\mathtt{x}\mathtt{y}c_2) = -\eta(c_1) + \eta(\mathtt{x}) + \eta(\mathtt{y}) + \eta(c_2) \\
\eta(x_2) &= \eta(c_2^{-1}\mathtt{y}^{-1}\mathtt{x}c_3) = -\eta(c_2) - \eta(\mathtt{y}) + \eta(\mathtt{x}) + \eta(c_3) \\
\eta(x_3) &= \eta(c_3^{-1}\mathtt{x}^{-1}c_1) = -\eta(c_3) - \eta(\mathtt{x}) + \eta(c_1)
\end{aligned}$$

  and hence

$$|\eta(x_1) + \eta(x_2) + \eta(x_3)| = |\eta(\mathtt{x})| = 1$$

  for any $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$.

- Type [T1b]: $(c_1^{-1}\mathtt{y}\mathtt{x}c_2, c_2^{-1}\mathtt{x}^{-1}\mathtt{y}c_3, c_3^{-1}\mathtt{y}^{-1}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. This case is analogous to the previous case.

- Type [T2a]: $(c_1^{-1}\mathtt{y}^{-1}\mathtt{x}\mathtt{y}c_2, c_2^{-1}\mathtt{y}^{-1}, \mathtt{y}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. Again we calculate

$$\begin{aligned}
\eta(x_1) &= \eta(c_1^{-1}\mathtt{y}^{-1}\mathtt{x}\mathtt{y}c_2) = -\eta(c_1) - \eta(\mathtt{y}) + \eta(\mathtt{x}) + \eta(\mathtt{y}) + \eta(c_2) \\
\eta(x_2) &= \eta(c_2^{-1}\mathtt{y}^{-1}) = -\eta(c_2) - \eta(\mathtt{y}) \\
\eta(x_3) &= \eta(\mathtt{y}c_1) = \eta(\mathtt{y}) + \eta(c_1)
\end{aligned}$$

and hence again

$$|\eta(x_1) + \eta(x_2) + \eta(x_3)| = |\eta(\mathtt{x})| = 1$$

for any $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$.

- Type [T2b]: $(c_1^{-1}\mathtt{x}^{-1}\mathtt{y}\mathtt{x}\mathtt{b}c_2, c_2^{-1}\mathtt{x}^{-1}, \mathtt{x}c_1)$, for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. This case is analogous to the previous case.

□

5.2 Gaps via Letter-Quasimorphisms

The aim of this section is to define letter-quasimorphisms and deduce the criterion for 1/2 gaps in scl. There will be two types of letter-quasimorphisms: *(general) letter-quasimorphisms* (Definition 5.2.1) and *well-behaved letter-quasimorphisms* (Definition 5.2.3). The former is useful for applications, the latter will be useful for proofs. For each letter-quasimorphism $\Phi\colon G \to \mathcal{A}$ there will be an associated well-behaved letter-quasimorphism $\tilde{\Phi}\colon G \to \mathcal{A}$ where $\tilde{\Phi}(g)$ is obtained from $\Phi(g)$ by modifying its beginning and its end; see Proposition 5.2.5.

5.2.1 (Well-Behaved) Letter-Quasimorphisms

As always $\mathcal{A}$ denotes the set of alternating words of $\mathbb{F}_2$ in the generators $\mathtt{a}$ and $\mathtt{b}$.

Definition 5.2.1. Let G be a group. We say that $\Phi\colon G \to \mathcal{A}$ is a *letter-quasimorphism* if Φ is alternating, i.e. $\Phi(g^{-1}) = \Phi(g)^{-1}$ for every $g \in G$ and if for every $g, h \in G$ one of the following holds:

1. $\Phi(g)\Phi(h)\Phi(gh)^{-1} = e$, or
2. there are elements $c_1, c_2, c_3 \in \mathcal{A}$ and letters $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3$ such that either $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3 \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and $\mathtt{x}_1\mathtt{x}_2\mathtt{x}_3 \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ or $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3 \in \{\mathtt{b}, \mathtt{b}^{-1}\}$ and $\mathtt{x}_1\mathtt{x}_2\mathtt{x}_3 \in \{\mathtt{b}, \mathtt{b}^{-1}\}$ which satisfy that $\Phi(g) = c_1^{-1}\mathtt{x}_1 c_2$, $\Phi(h) = c_2^{-1}\mathtt{x}_2 c_3$ and $\Phi(gh)^{-1} = c_3^{-1}\mathtt{x}_3 c_1$ as freely reduced alternating words.

The motivating example for letter-quasimorphisms is the following:

Example 5.2.2. Consider the map $\Phi\colon \mathbb{F}_2 \to \mathcal{A}$ defined as follows. Suppose that $w \in \mathbb{F}_2$ has reduced representation $\mathtt{a}^{n_1}\mathtt{b}^{m_1}\cdots\mathtt{a}^{n_k}\mathtt{b}^{m_k}$ with all $n_i, m_i \in \mathbb{Z}$ where all but possibly n_1 and / or m_k are non-zero. Then set

$$\Phi(w) = \mathtt{a}^{\text{sign}(n_1)}\mathtt{b}^{\text{sign}(m_1)}\cdots\mathtt{a}^{\text{sign}(n_k)}\mathtt{b}^{\text{sign}(m_k)}$$

where $\text{sign}\colon \mathbb{Z} \to \{+1, 0, -1\}$ is defined as usual. This may be seen to be a letter-quasimorphism and will be vastly generalised to amalgamated free products; see Lemma 5.4.1. Observe that for any group G and any homomorphism $\Omega\colon G \to \mathbb{F}_2$ the map $\Phi\circ\Omega\colon G \to \mathcal{A}$ is a letter-quasimorphism. Suppose that G is *residually free*. Then for every non-trivial element $g \in G$ there is a homomorphism $\Omega_g\colon G \to \mathbb{F}_2$ such that $\Omega_g(g) \in \mathbb{F}_2$ is nontrivial. By applying a suitable automorphism on $\mathbb{F}_2$ to Ω_g we may assume that $\Omega_g(g)$ starts in a power of $\mathtt{a}$ and ends in a power of $\mathtt{b}$. Then $\Phi_g := \Phi \circ \Omega_g$ is a letter quasimorphism such that $\Phi_g(g)$ is nontrivial and such that $\Phi_g(g^n) = \Phi_g(g)^n$.

Definition 5.2.3. We will call triples (x_1, x_2, x_3) *degenerate* if they are equivalent to a triple (w, w^{-1}, e) for some $w \in \mathcal{A}$.

Let G be a group. A map $\Psi\colon G \to \mathcal{A}$ is called *well-behaved letter-quasimorphism* if Ψ is alternating, i.e. $\Psi(g^{-1}) = \Psi(g)^{-1}$ for every $g \in G$, and for all $g, h \in G$, the triple

$$(\Psi(g), \Psi(h), \Psi(gh)^{-1})$$

is either letter-thin (see Definition 5.1.12) or degenerate.

Remark 5.2.4. Note that a triple (x_1, x_2, x_3) is degenerate if and only if there is some $w \in \mathcal{A}$ such that (x_1, x_2, x_3) equals (w, w^{-1}, e), (w, e, w^{-1}) or (e, w, w^{-1}). Note that if $\Phi\colon G \to \mathcal{A}$ is a well-behaved letter-quasimorphism then also $\alpha \circ \Phi\colon G \to \mathcal{A}$ and $\beta \circ \Phi\colon G \to \mathcal{A}$ are well-behaved letter-quasimorphisms. This follows immediately from Lemma 5.1.14 and the fact that α (resp. β) satisfies $\alpha(w^{-1}) = \alpha(w)^{-1}$ (resp. $\beta(w^{-1}) = \beta(w)^{-1}$) for any $w \in \mathcal{A}$.

It is easy to see that every well-behaved letter-quasimorphism is also a letter-quasimorphism. The contrary does not hold. The map $\Phi\colon \mathbb{F}_2 \to \mathcal{A}$ described in Example 5.2.2 is a letter-quasimorphism but not a well-behaved letter-quasimorphism.

For example for $g = \mathtt{a}$, $h = \mathtt{a}$ we obtain $(\Phi(g), \Phi(h), \Phi(h^{-1}g^{-1})) = (\mathtt{a}, \mathtt{a}, \mathtt{a}^{-1})$, which is neither letter-thin nor degenerate.

However, we may assign to each letter-quasimorphism Φ a well-behaved letter-quasimorphism $\tilde{\Phi}$. This will be done by pre-composing Φ with a map $w \mapsto \tilde{w}$ defined as follows.

Set $\tilde{w} = e$ whenever $w \in \{\mathtt{a}, e, \mathtt{a}^{-1}\}$. Else let $\mathtt{z}_s$ be the first and $\mathtt{z}_e$ be the last letter of $w \in \mathcal{A}$. Define $\tilde{w}$ as the reduced element in $\mathbb{F}_2$ freely equal to $\tilde{w} := \zeta_s(\mathtt{z}_s) w \zeta_e(\mathtt{z}_e)$ where

$$\zeta_s(\mathtt{z}) = \begin{cases} e & \text{if } \mathtt{z} = \mathtt{a} \\ \mathtt{a} & \text{if } \mathtt{z} = \mathtt{b} \text{ or } \mathtt{b}^{-1} \\ \mathtt{a}^2 & \text{if } \mathtt{z} = \mathtt{a}^{-1} \end{cases}$$

and

$$\zeta_e(\mathtt{z}) = \begin{cases} e & \text{if } \mathtt{z} = \mathtt{a}^{-1} \\ \mathtt{a}^{-1} & \text{if } \mathtt{z} = \mathtt{b} \text{ or } \mathtt{b}^{-1} \\ \mathtt{a}^{-2} & \text{if } \mathtt{z} = \mathtt{a}. \end{cases}$$

The key point is that $\tilde{w}$ starts with $\mathtt{a}$ and ends with $\mathtt{a}^{-1}$, unless $w \in \{\mathtt{a}, e, \mathtt{a}^{-1}\}$. Observe that $\zeta_e(\mathtt{z})^{-1} = \zeta_s(\mathtt{z})$, and hence the map $w \mapsto \tilde{w}$ is alternating, i.e. $\widetilde{w^{-1}} = \tilde{w}^{-1}$. For example, $\mathtt{a} \mapsto e$, $\mathtt{aba}^{-1} \mapsto \mathtt{aba}^{-1}$ and $\mathtt{a}^{-1}\mathtt{baba} \mapsto \mathtt{ababa}^{-1}$.

If $\Phi \colon G \to \mathcal{A}$ is a letter-quasimorphism then we define $\tilde{\Phi} \colon G \to \mathcal{A}$ via $\tilde{\Phi}(g) := \widetilde{\Phi(g)}$.

Proposition 5.2.5. *If $\Phi \colon G \to \mathcal{A}$ is a letter-quasimorphism then $\tilde{\Phi} \colon G \to \mathcal{A}$ is a well-behaved letter-quasimorphism, called the* associated *well-behaved letter-quasimorphism.*

Proof. As $w \mapsto \tilde{w}$ commutes with taking inverses, if Φ is alternating then so is $\tilde{\Phi}$. In what follows we will use the following easy to check claim.

Claim 5.2.6. *Let (x_1, x_2, x_3) be an arbitrary triple obtained from (y_1, y_2, y_3) by applying a sequence of the equivalences (i) and (ii) of Definition 5.1.10. Then $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \sim (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. In this case we say that the triples (x_1, x_2, x_3) and (y_1, y_2, y_3) are* equivalent up to rotation and inverses*.*

Let $g, h \in G$. We wish to show that $(\tilde{\Phi}(g), \tilde{\Phi}(h), \tilde{\Phi}(gh)^{-1})$ is a letter-thin triple or degenerate, i.e. equivalent to (w, w^{-1}, e) for some $w \in \mathcal{A}$. If $(\Phi(g), \Phi(h), \Phi(gh)^{-1})$ is equivalent up to rotation and inverses to (u_1, u_2, u_3) the above claim implies that it suffices to check that $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ is either letter-thin or equivalent to (w, w^{-1}, e).

First suppose that g, h are as in Case (1) of Definition 5.2.1 i.e. $\Phi(g)\Phi(h)\Phi(gh)^{-1} = e$. If one of $\Phi(g)$, $\Phi(h)$ and $\Phi(gh)$ are trivial then the two other elements are inverses. Hence, up to rotation and taking inverses we may assume that

$$(\Phi(g), \Phi(h), \Phi(gh)^{-1}) = (u, u^{-1}, e)$$

for some $u \in \mathcal{A}$. Hence $(\tilde{u}, \tilde{u}^{-1}, e)$ is degenerate.

If none of $\Phi(g)$, $\Phi(h)$ and $\Phi(gh)^{-1}$ are trivial then, as Φ maps to alternating elements, there are elements u_1, u_2 such that u_1 ends in a power of $\mathtt{a}$ and u_2 starts in a power of $\mathtt{b}$, such that $(\Phi(g), \Phi(h), \Phi(gh))$ is equivalent up to rotation and taking inverses to (u_1, u_2, u_3) where $u_3 = u_2^{-1}u_1^{-1}$ as a reduced word. Further, write $u_1 = u_1'\mathtt{x}$ as a reduced word for $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$ and an appropriate word $u_1' \in \mathcal{A}$. If u_1' is empty, then $\tilde{u}_1 = e$. Let $\mathtt{z}_2$ be the last letter of u_2. Then

$$(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (e, \mathtt{a}u_2\zeta_e(\mathtt{z}_2), \zeta_e(\mathtt{z}_2)^{-1}u_2^{-1}\mathtt{a}^{-1})$$

which is equivalent to (w, w^{-1}, e) for $w = \mathtt{a}u_2\zeta_e(\mathtt{z}_2)$. If u_1' is non-empty, let $\mathtt{z}_1$ be the first letter of u_1' and as before let $\mathtt{z}_2$ be the last letter of u_2. Then

$$(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (\zeta_s(\mathtt{z}_1)u_1'\mathtt{a}^{-1}, \mathtt{a}u_2\zeta_e(\mathtt{z}_2), \zeta_e(\mathtt{z}_2)^{-1}u_2^{-1}\mathtt{x}^{-1}u_1'^{-1}\zeta_s(\mathtt{z}_1)^{-1})$$

which can be seen to be letter-thin of type [T1$\mathtt{a}$]. This shows that $(\tilde{\Phi}(g), \tilde{\Phi}(h), \tilde{\Phi}(gh)^{-1})$ is letter-thin or degenerate if $\Phi(g)\Phi(h)\Phi(gh)^{-1} = e$.

Hence, suppose that g, h are as in Case (2) of Definition 5.2.1. Then $(\Phi(g), \Phi(h), \Phi(gh))$ is equivalent up to rotation and inverses to

$$(u_1, u_2, u_3) = (c_1^{-1}\mathtt{x}c_2, c_2^{-1}\mathtt{x}c_3, c_3^{-1}\mathtt{x}^{-1}c_1)$$

for $\mathtt{x} \in \{\mathtt{a}, \mathtt{b}\}$ where $c_1, c_2, c_3 \in \mathcal{A}$ are *arbitrary* i.e. we do not assume that c_2 is non-empty as in Definition 5.1.12. First, suppose that $\mathtt{x} = \mathtt{b}$. Define

$$d_i = \begin{cases} c_i\zeta_e(\mathtt{z}_i) & \text{if } c_i \neq e \\ \mathtt{a}^{-1} & \text{else} \end{cases}$$

where $\mathtt{z}_i$ is the last letter of c_i. We may see then, that

$$(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (d_1^{-1}\mathtt{b}d_2, d_2^{-1}\mathtt{b}d_3, d_3^{-1}\mathtt{b}^{-1}d_1)$$

which is letter thin of type [T1$\mathtt{b}$] as all d_i's are non trivial.

Hence, suppose that $\mathtt{x} = \mathtt{a}$. For what follows, if c_i is non-empty, we will denote by $\mathtt{z}_i$ the last letter of c_i and let d_i be the freely reduced word represented by $c_i\zeta_e(\mathtt{z}_i)$. Observe that if c_i is non-empty then so is d_i.

There are the following cases:

(i) $c_1 \neq e$, $c_2 \neq e$, $c_3 \neq e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (d_1^{-1}\mathtt{a}d_2, d_2^{-1}\mathtt{a}d_3, d_3^{-1}\mathtt{a}^{-1}d_1)$

(ii) $c_1 \neq e$, $c_2 \neq e$, $c_3 = e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (d_1^{-1}\mathtt{a}d_2, d_2^{-1}\mathtt{a}^{-1}, \mathtt{a}d_1)$

(iii) $c_1 \neq e$, $c_2 = e$, $c_3 \neq e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (d_1^{-1}\mathtt{a}^{-1}, \mathtt{a}d_3, d_3^{-1}\mathtt{a}^{-1}d_1)$

(iv) $c_1 = e$, $c_2 \neq e$, $c_3 \neq e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (\mathtt{a}d_2, d_2^{-1}\mathtt{a}d_3, d_3^{-1}\mathtt{a}^{-1})$

(v) $c_1 \neq e$, $c_2 = e$, $c_3 = e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (d_1^{-1}\mathtt{a}, e, \mathtt{a}^{-1}d_1)$

(vi) $c_1 = e$, $c_2 \neq e$, $c_3 = e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (\mathtt{a}d_2, d_2^{-1}\mathtt{a}^{-1}, e)$

(vii) $c_1 = e$, $c_2 = e$, $c_3 \neq e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (e, \mathtt{a}d_3, d_3^{-1}\mathtt{a}^{-1})$

(viii) $c_1 = e$, $c_2 = e$, $c_3 = e$: Then $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = (e, e, e)$

and cases $(i)-(iv)$ can be seen to be letter-thin of type [T1$\mathtt{a}$] and cases $(v)-(viii)$ can be seen to be degenerate. This completes the proof. □

Both letter-quasimorphisms and well-behaved letter-quasimorphisms are examples of *quasimorphism* in the sense of Hartnick–Schweitzer [HS16]; see Subsection 2.2.6. Let Φ be a letter-quasimorphism and let $\bar{\eta}\colon \mathbb{F}_2 \to \mathbb{R}$ be an ordinary homogeneous quasimorphism with defect D which vanishes on the generators $\mathtt{a}, \mathtt{b}$. We wish to calculate the defect of $\bar{\eta} \circ \Phi$. Fix $g, h \in G$. If $\Phi(g)\Phi(h) = \Phi(gh)$, then

$$|\bar{\eta} \circ \Phi(g) + \bar{\eta} \circ \Phi(h) - \bar{\eta} \circ \Phi(gh)| \leq D$$

Else, up to rotating the factors we see that

$$(\Phi(g), \Phi(h), \Phi(gh)^{-1}) = (d_1^{-1}\mathtt{x}d_2, d_2^{-1}d_3, d_3^{-1}d_1)$$

for some appropriate $d_1, d_2, d_3 \in \mathcal{A}$, $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}, \mathtt{b}, \mathtt{b}^{-1}\}$. Then, as $\bar{\eta}$ is homogeneous $\bar{\eta}(d_1^{-1}\mathtt{x}d_2) = \bar{\eta}(\mathtt{x}d_2d_1^{-1})$ and hence $|\bar{\eta}(\mathtt{x}d_2d_1^{-1}) - \bar{\eta}(d_2d_1^{-1})| \leq D$ as we assumed that $\bar{\eta}$ vanishes on the generators. Then we may estimate

$$|\bar{\eta} \circ \Phi(g) + \bar{\eta} \circ \Phi(h) + \bar{\eta} \circ \Phi(gh)^{-1}| = |\bar{\eta}(d_1^{-1}\mathtt{x}d_2) + \bar{\eta}(d_2^{-1}d_3) + \bar{\eta}(d_3^{-1}d_1)| \leq 4D$$

and after homogenisation of $\phi = \bar{\eta} \circ \Phi(g)$ we estimate that $D(\bar{\phi}) \leq 8D$ using that homogenisation at most doubles the defect; see Proposition 2.2.5. Hence if $\Phi(g) \in \mathbb{F}_2'$ is such that $\Phi(g^n) = w^n$ for some non-trivial $w \in \mathcal{A}$ which also lies in the commutator subgroup $\mathbb{F}'$ and $\eta \colon \mathbb{F}_2 \to \mathbb{R}$ is homogenous and extremal to $\Phi(g)$ with defect 1 then, by Bavard,

$$\operatorname{scl}(g) \geq \frac{\bar{\phi}(g)}{16} \geq \frac{\bar{\eta}(\Phi(g))}{16} = \frac{\operatorname{scl}(\Phi(g))}{8}$$

and in particular $\operatorname{scl}(g) \geq 1/16$. This is already a good estimate but we see that we can do much better; see Theorem E.

We will see that this notion is much more flexible than homomorphisms. There are groups G such that for every non-trivial element $g \in G'$ there is a letter-quasimorphisms Φ such that $\Phi(g)$ is non-trivial. This may be possible even if the group G is not residually free, for example if G is a right-angled Artin group; see Section 5.5.

5.2.2 Main Theorem

We now deduce our main criterion for 1/2-gaps in scl:

Theorem E. *Let G be a group and let $g_0 \in G$. Suppose there is a letter-quasimorphism $\Phi \colon G \to \mathcal{A}$ such that $\Phi(g_0)$ is non-trivial and that $\Phi(g_0^n) = \Phi(g_0)^n$ for all $n \in \mathbb{N}$. Then there is an explicit homogeneous quasimorphism $\bar{\phi} \colon G \to \mathbb{R}$ with $D(\bar{\phi}) \leq 1$ such that $\bar{\phi}(g_0) \geq 1$. If $g_0 \in G'$, then scl$(g_0) \geq 1/2$.*

If G is countable then there is an action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1\bar{\phi}] = \rho^*\mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}_b^2(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.

In particular, the $\Phi(g_0) \in \mathcal{A}$ of the Theorem has to be alternating and of *even length*, else $\Phi(g_0)^n$ would not be an alternating word.

Proof. Let $\Phi \colon G \to \mathcal{A}$ be the letter-quasimorphism as in the theorem and let $\tilde{\Phi} \colon G \to \mathcal{A}$ be the associated well behaved letter-quasimorphism described above. As $\tilde{\Phi}(g_0)$ is obtained from $\Phi(g_0)$ by just possibly changing the beginning and the end of the word $\Phi(g_0)$, it is easy to see that there are words $c_1, c_2, w \in \mathcal{A}$ such that $\tilde{\Phi}(g_0^n) = c_1^{-1} w^{n-1} c_2$ as a freely reduced word for all $n \geq 1$.

Consider the sequence γ_i of maps $\gamma_i \colon \mathcal{A} \to \mathcal{A}$ defined via $\gamma_0 = id$, $\gamma_{2k+1} = (\alpha \circ \beta)^k \circ \alpha$ and $\gamma_{2k} = (\beta \circ \alpha)^k$ and note that γ_i is either $\alpha \circ \gamma_{i-1}$ or $\beta \circ \gamma_{i-1}$; see Definition 5.1.2. Analogously define the sequence $\bar{\gamma}_i \colon \bar{\mathcal{A}}_0 \to \bar{\mathcal{A}}_0$ of maps via $\bar{\gamma}_0 = id$, $\bar{\gamma}_{2k+1} = (\bar{\alpha} \circ \bar{\beta})^k \circ \bar{\alpha}$ and $\bar{\gamma}_{2k} = (\bar{\beta} \circ \bar{\alpha})^k$ and note that every $\bar{\gamma}_i$ is either $\bar{\alpha} \circ \bar{\gamma}_{i-1}$ or $\bar{\beta} \circ \bar{\gamma}_{i-1}$; see Definition 5.1.6. For every letter-thin triple (x_1, x_2, x_3) also $\gamma_i(x_1, x_2, x_3)$ is letter-thin by multiple applications of Lemma 5.1.14. Furthermore, if (x_1, x_2, x_3) is a degenerate triple as in Definition 5.2.3, then also $\gamma_i(x_1, x_2, x_3)$ is a degenerate triple as γ_i satisfies $\gamma_i(x^{-1}) = \gamma_i(x)^{-1}$ for all $x \in \mathcal{A}$.

Let w be as above and consider the sequence $\bar{\gamma}_i(w) \in \bar{\mathcal{A}}_0$ of conjugacy classes in $\bar{\mathcal{A}}_0$. By Proposition 5.1.8, if $\bar{\gamma}_i(w)$ is a non-trivial equivalence class in the commutator subgroup then $\bar{\gamma}_{i+1}(w)$ either is non-trivial and has strictly smaller word-length or $\bar{\gamma}_i(w) = \bar{\gamma}_{i+1}(w)$; see also Remark 5.1.5.

Hence, there are the following cases:

- For all $i \in \mathbb{N}$, $\bar{\gamma}_i(w)$ lies in $\mathbb{F}_2'$, the commutator subgroup. Then, there is an N such that $\bar{\gamma}_N(w) = \bar{\gamma}_{N+i}(w)$ for all $i \in \mathbb{N}$. Both $\bar{\alpha}$ and $\bar{\beta}$ then fix the class $\bar{\gamma}_N(w)$. By Proposition 5.1.8, $\bar{\gamma}_N(w)$ may be represented by $[\mathtt{a}, \mathtt{b}]^k$ for $k \in \mathbb{Z} \backslash \{0\}$. Hence, the quasimorphism $\eta_0 = \eta_{\mathtt{ab}} - \eta_{\mathtt{ba}}$ studied in Example 2.3.3 and Proposition 5.1.15, satisfies that $|\bar{\eta}_0(\bar{\gamma}_N(w))| \geq 2$. Define $\psi \colon G \to \mathbb{Z}$ via
$$\psi(g) := \begin{cases} \eta_0 \circ \gamma_N \circ \tilde{\Phi}(g) & \text{if } \gamma_N \circ \tilde{\Phi}(g) \neq e \\ 1 & \text{else} \end{cases}$$
and observe that if $\gamma_N \circ \tilde{\Phi}(g)$ is non-trivial, then $\psi(g^{-1}) = -\psi(g)$. By multiple applications of Proposition 5.1.9, we see that there are some elements $d_1, d_2, w' \in \mathcal{A}$ such that $\gamma_N \circ \tilde{\Phi}(g^n) = d_1 w'^{n-K} d_2$ for all $n \geq K$, for $K \leq N+1$

and $[w'] = \bar{\gamma}_N([w])$. We see that

$$\begin{aligned}|\bar{\psi}(g_0)| &= \lim_{n\to\infty} |\psi(g_0^n)|/n \\ &= \lim_{n\to\infty} |\eta_0 \circ \gamma_N \circ \tilde{\Phi}(g_0^n)|/n \\ &= \lim_{n\to\infty} |\eta_0(d_1 w'^{n-K} d_2)|/n \\ &= |\bar{\eta_0}(\bar{\gamma}_N([w]))| \geq 2.\end{aligned}$$

By multiple applications of Lemma 5.1.14 and the fact that $\alpha(w^{-1}) = \alpha(w)^{-1}$, $\beta(w^{-1}) = \beta(w)^{-1}$ and $\alpha(e) = e = \beta(e)$ we see that $\gamma_N \circ \tilde{\Phi}$ is a well-behaved letter-quasimorphism. Let $g, h \in G$. We wish to compute the defect $|\psi(g)+\psi(h)-\psi(gh)|$. To ease notation define (x_1, x_2, x_3) as the triple

$$(x_1, x_2, x_3) = (\gamma_N \circ \tilde{\Phi}(g), \gamma_N \circ \tilde{\Phi}(h), \gamma_N \circ \tilde{\Phi}(gh)^{-1})$$

which is either letter-thin or degenerate as $\gamma_N \circ \tilde{\Phi}$ is a well-behaved letter-quasimorphism. If (x_1, x_2, x_3) letter-thin then none of its components x_i are empty. Hence by Proposition 5.1.15,

$$\begin{aligned}|\psi(g) + \psi(h) - \psi(gh)| =& |\psi(g) + \psi(h) + \psi(h^{-1}g^{-1})| \\ =& |\eta_0(x_1) + \eta_0(x_2) + \eta_0(x_3)| \\ =& 1.\end{aligned}$$

Suppose that (x_1, x_2, x_3) is degenerate. Then one may see that (x_1, x_2, x_3) equals (v, v^{-1}, e), (v, e, v^{-1}) or (e, v, v^{-1}) for some $v \in \mathcal{A}$. Using that $-\eta_0(v) = \eta_0(v^{-1})$ for $e \neq v \in \mathcal{A}$ we see that two terms of $\psi(g) + \psi(h) - \psi(gh)$ will cancel and for the other will be 1. Hence, $|\psi(g)+\psi(h)-\psi(gh)| = 1$. Finally, if $(x_1, x_2, x_3) = (e, e, e)$ then $\psi(g) + \psi(h) - \psi(gh) = 1$. In particular we see that for any $g, h \in G$, $\psi(g) + \psi(h) - \psi(gh) \in \{1, -1\}$, so ψ is a quasimorphism. Moreover, by possibly changing the sign of ψ we may assume that $\bar{\psi}(g_0) \geq 2$.

- Otherwise, let $N \in \mathbb{N}$ be the smallest integer such that $\bar{\gamma}_N(w) \notin \mathbb{F}_2'$. Then $\bar{\gamma}_N(w) \in \mathcal{A}$ is represented by a non-trivial even word which is not in the commutator. Hence

$$|\eta_{\mathtt{a}}(\bar{\gamma}_N(w))| + |\eta_{\mathtt{b}}(\bar{\gamma}_N(w))| \geq 2$$

where $\eta_{\mathtt{a}}\colon \mathbb{F}_2 \to \mathbb{Z}$ (resp. $\eta_{\mathtt{b}}\colon \mathbb{F}_2 \to \mathbb{Z}$) denotes the homomorphism counting the letter $\mathtt{a}$ (resp. $\mathtt{b}$). Observe that homomorphisms are already homogenised. There is some $\eta = \eta_{\mathtt{x}} + \eta_{\mathtt{y}}$ where $\mathtt{x} \in \{\mathtt{a}, \mathtt{a}^{-1}\}$, $\mathtt{y} \in \{\mathtt{b}, \mathtt{b}^{-1}\}$ such that $\eta(\bar{\gamma}_N(w)) \geq 2$. As before, define $\psi\colon G \to \mathbb{Z}$ via

$$\psi(g) := \begin{cases} \eta \circ \gamma_N \circ \tilde{\Phi}(g) & \text{if } \gamma_N \circ \tilde{\Phi}(g) \neq e \\ 1 & \text{else.} \end{cases}$$

By a similar argument as above we see that $\bar{\psi}(g_0) \geq 2$. Again, the triple

$$(x_1, x_2, x_3) = (\gamma_N \circ \tilde{\Phi}(g), \gamma_N \circ \tilde{\Phi}(h), \gamma_N \circ \tilde{\Phi}(h^{-1}g^{-1}))$$

is either letter-thin or degenerate. By the same argument as in the previous case and using Proposition 5.1.16 we conclude that for any $g, h \in G$, $|\psi(g) + \psi(h) - \psi(gh)| = 1$, so ψ is a quasimorphism. In particular we see that for any $g, h \in G$, $\psi(g) + \psi(h) - \psi(gh) \in \{1, -1\}$.

In both cases, set

$$\phi(g) := \frac{\psi(g) + 1}{2}.$$

Then we see that, for any $g, h \in G$,

$$\delta^1\phi(g, h) = \phi(g) + \phi(h) - \phi(gh) = \frac{\psi(g) + \psi(h) - \psi(gh) + 1}{2} \in \{0, 1\}.$$

Hence, by Theorem 2.2.4 due to Ghys (see also [Ghy87]), there is an action $\rho\colon G \to \text{Homeo}^+(S^1)$ on the circle such that $\rho^*\text{eu}_b = [\delta^1\phi] \in \text{H}^2_b(G, \mathbb{Z})$ and hence $\rho^*\text{eu}^{\mathbb{R}}_b = [\delta^1\bar{\phi}] \in \text{H}^2_b(G, \mathbb{R})$. Here, eu_b (resp. $\text{eu}^{\mathbb{R}}_b$) denotes the (real) bounded Euler class. Moreover, we observe that $\bar{\phi}(g) = \bar{\psi}(g)/2$, for $\bar{\phi}$ the homogenisation of ϕ. Furthermore, as $D(\psi) = 1$ we estimate by Proposition 2.2.5 that $D(\bar{\psi}) \leq 2$ and hence $D(\bar{\phi}) \leq 1$.

We conclude that there is a quasimorphism $\phi\colon G \to \mathbb{R}$ with homogenisation $\bar{\phi}$ such that $D(\bar{\phi}) \leq 1$, $\bar{\phi}(g_0) \geq 1$. If G is countable then there is an action $\rho\colon G \to \text{Homeo}^+(S^1)$ with $[\delta^1\phi] = \rho^*\text{eu}^{\mathbb{R}}_b \in \text{H}^2_b(G, \mathbb{R})$ where $\text{eu}^{\mathbb{R}}_b$ is the real bounded Euler class. □

Applying Theorem E to Example 5.2.2 we recover that in every residually free group G, every non-trivial element $g \in G'$ has stable commutator length at least $1/2$. This gap is realised by a quasimorphism induced by a circle action which has not been known previously.

As said in the introduction we think of letter-quasimorphisms as simplifications of elements. Sometimes information about w can not be recovered by $\Phi(w)$. For example for the word $w = \mathtt{aba}^{-1}\mathtt{b}^{-1}\mathtt{ab}^{-3}\mathtt{a}^{-1}\mathtt{b}^{3}$, we may compute[1] $\mathrm{scl}(w) = 3/4$ but $\mathrm{scl}(\Phi(w)) = 1/2$. This example may be generalised: Pick an alternating word $w \in \mathcal{A}$ that starts and ends in a power of $\mathtt{b}$. Then $[\mathtt{a}, w] \in \mathcal{A}$ and $\mathrm{scl}([\mathtt{a}, w]) = 1/2$. Then for any choice of words $v_1, v_2 \in \mathbb{F}_2$ such that $\Phi(v_1) = w$, $\Phi(v_2) = w^{-1}$ and such that $v = \mathtt{a}v_1\mathtt{a}^{-1}v_2 \in \mathbb{F}_2'$ we have that $\Phi(v) = [\mathtt{a}, w]$. However, $\mathrm{scl}(v)$ is experimentally arbitrarily large.

Remark 5.2.7. As pointed out in the proof all of $\gamma_i \circ \tilde{\Phi}$ are well-behaved letter-quasimorphisms for any $i \in \mathbb{N}$. The quasimorphisms ψ defined in the proof are then pullbacks of the quasimorphism $\eta_0 = \eta_{\mathtt{ab}} - \eta_{\mathtt{ba}}$ or homomorphisms $\eta = \eta_{\mathtt{x}} + \eta_{\mathtt{y}}$ via these well-behaved letter-quasimorphisms $\gamma_i \circ \tilde{\Phi} \colon G \to \mathcal{A} \subset \mathbb{F}_2$.

Remark 5.2.8. In light of Theorem 2.3.2, a criterion for groups to have the optimal scl-gap of $1/2$ may hence be as follows:

Let G be a non-abelian group. If for every non-trivial element $g \in G'$ there is a letter-quasimorphism $\Phi \colon G \to \mathcal{A}$ such that $\Phi(g^n) = \Phi(g)^n$ where $\Phi(g)$ is non-trivial. Then G has a gap of $1/2$ in stable commutator length.

By Example 5.2.2 residually free groups have this property and the criterion has some qualitative similarities to being residually free. We will later see that also non-residually free groups, like right-angled Artin groups, have this property; see Section 5.5.

5.3 Left Orders and Left-Relatively Convex Subgroups

For what follows we will use the notation and conventions of [ADS15]. We further emphasise that nothing in this section is original work.

[1] These calculations are done with `scallop`, see [Cal]

An order $\prec$ on a set $\mathcal{X}$ is a subset of $\mathcal{X} \times \mathcal{X}$ where we stress that a pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ is in this subset by writing $x \prec y$. Furthermore, the following holds:

- For all $x, y \in \mathcal{X}$ either $x \prec y$ or $y \prec x$. We have $x \prec y$ and $y \prec x$ if and only if $x = y$.
- For all $x, y, z \in \mathcal{X}$ such that $x \prec y$ and $y \prec z$ we have $x \prec z$.

A set $\mathcal{X}$ with a left group action has a *G-invariant order* if for all $g \in G$, $x_1, x_2 \in \mathcal{X}$, $x_1 \prec x_2$ implies that $g.x_1 \prec g.x_2$. A group G is said to be *left orderable* if the set G has a G-invariant order with respect to its left action on itself. A subgroup $H < G$ is said to be *left relatively convex* in G if the G-set G/H has some G-invariant order. Note that this definition is valid even if G itself is *not* left-orderable. If G itself is orderable, then this is equivalent to the following: There is an order $\prec$ on G such that for every $h_1, h_2 \in H$ and $g \in G$ with $h_1 \prec g \prec h_2$ we may conclude $g \in H$. In this case we simply say that H is convex in G. As $e \in H$, this means that H is a neighbourhood of e. It is not hard to see that left relatively convex is transitive:

Proposition 5.3.1. [2] *Let $K < H < G$ be groups. Then G/K is G-orderable such that H/K is convex if and only if G/H is G-orderable and H/K is H-orderable.*

An easy example of a pair $H < G$ such that H is left relatively convex in G is $\mathbb{Z} < \mathbb{Z}^2$ embedded in the second coordinate via the standard lexicographic order. Similarly, every subgroup $G < \mathbb{Z} \times G$ embedded via the second coordinate, is left relatively convex for an arbitrary group G. Every generator of a non-abelian free group generates a left relatively convex subgroup in the total group; see [DH91]. In fact, [ADS15] show that each maximal cyclic subgroup of a right-angled Artin group is left relatively convex.

We wish to state the main Theorem of [ADS15]. For this let T denote an *oriented* simplicial tree, with vertices V(T) and edges E(T) and two maps $\iota, \tau \colon \mathrm{E(T)} \to \mathrm{V(T)}$ assigning to each oriented edge its initial and terminal vertex respectively. Suppose that G acts on T and denote by G_v (resp. G_e) the stabilisers of a vertex

[2] See Section 2 of [ADS15]

$v \in \mathrm{V(T)}$ (resp. an edge $e \in \mathrm{E(T)}$). Note that stabilisers of an edge e naturally embed into $G_{\iota(e)}$ and $G_{\tau(e)}$.

Theorem 5.3.2. [3] *Suppose that* T *is a left* G*-tree such that, for each* T*-edge* e*,* G_e *is left relatively convex in* $G_{\iota(e)}$ *and in* $G_{\tau(e)}$*. Then, for each* $v \in \mathrm{V(T)}$*,* G_v *is left relatively convex in* G*. Moreover, if there exists some* $v \in \mathrm{V(T)}$ *such that* G_v *is left orderable, then* G *is left orderable.*

We deduce the following corollary, see Example 19 of [ADS15] using Bass–Serre Theory.

Corollary 5.3.3. *Let* A, B *and* C *be groups and let* $\kappa_A \colon C \hookrightarrow A$ *and* $\kappa_B \colon C \hookrightarrow B$ *be injections and let* $G = A \star_C B$ *be the corresponding amalgamated free product (see Section 5.4). If* $\kappa_A(C)$ *is left relatively convex in* A *and* $\kappa_B(C)$ *is left relatively convex in* B*, then* A *and* B *are left relatively convex in* G*.*

Let $H < G$ be a left relatively convex subgroup and let $\prec$ be a G-invariant order of G/H. We define the *sign-function* $\mathrm{sign}\colon G \to \{-1, 0, 1\}$ on representatives $g \in G$ of cosets in G/H via

$$\mathrm{sign}(g) = \begin{cases} +1 & \text{if } gH \succ H \\ 0 & \text{if } g \in H \\ -1 & \text{if } gH \prec H \end{cases}$$

Proposition 5.3.4. *Let* $H < G$ *be a left relatively convex subgroup and let* $\prec$ *be the* G*-invariant order of* G/H*. Then the sign-function with respect to* $\prec$ *on elements in* G *is independent under left or right multiplication by elements of* H*. That is for every* $g \in G \setminus H$ *and for every* $h \in H$*, sign*$(hg) =$ *sign*$(g) =$ *sign*(gh)*.*

Proof. Clearly $\mathrm{sign}(gh) = \mathrm{sign}(g)$ as both g and gh define the same coset. On the other hand, if $hgH \succ H$ then by left multiplication $gH \succ H$ and similarly if $hgH \prec H$ then $gH \prec H$, so $\mathrm{sign}(hg) = \mathrm{sign}(g)$. □

[3] Theorem 14 of [ADS15]

5.4 Amalgamated Free Products

Let A, B, C be groups and let $\kappa_A \colon C \hookrightarrow A$, $\kappa_B \colon C \hookrightarrow B$ be injections. The *amalgamated free product* $G = A \star_C B$ with respect to κ_A and κ_B it the group via

$$G = A \star_C B = A \star B / \langle\langle \kappa_A(c)^{-1}\kappa_B(c) \mid c \in C \rangle\rangle.$$

It is a well-known fact that the homomorphism $A \to A \star_C B$ (resp. $B \to A \star_C B$) defined by mapping $a \in A$ (resp. $b \in B$) to the corresponding element $a \in G$ (resp. $b \in G$) is *injective* and that C embeds in G via these injections. See [Ser80] for a reference. Every element $g \in G$ with $g \in G \setminus C$ may be written as a product

$$g = d_1 \cdots d_k$$

such that all of d_i are either in $A \setminus \kappa_A(C)$ or in $B \setminus \kappa_B(C)$ and alternate between both. Furthermore for any other such expression

$$g = d'_1 \cdots d'_{k'}$$

one may deduce that $k' = k$ and that there are elements $c_i \in C$, $i \in \{1, \ldots, k-1\}$ such that $d'_1 = d_1 c_1$, $d'_i = c_{i-1}^{-1} d_i c_i$ and $d'_k = c_{k-1} d_k$.

For what follows, let $\prec_A$ (resp. $\prec_B$) be a left order on $A/\kappa_A(C)$ (resp. $B/\kappa_B(C)$) and let sign_A (resp. sign_B) be its sign on A (resp. B). We define the map $\Phi \colon G \to \mathcal{A}$ as follows: If $g \in C$ set $\Phi(g) = e$. Else let $g = d_1 \cdots d_k$ be the normal form described above. Then, set

$$\Phi(g) = \prod_{i=1}^{k} \Phi(d_i)$$

where we define

$$\Phi(d_i) = \begin{cases} \mathsf{a}^{\operatorname{sign}_A(d_i)} & \text{if } d_i \in A \setminus \kappa_A(C) \\ \mathsf{b}^{\operatorname{sign}_B(d_i)} & \text{if } d_i \in B \setminus \kappa_B(C) \end{cases}$$

and we note that Φ is well defined. To see this let $d'_1 \cdots d'_k$ be another normal form for g and let $c_i \in C$ for $i \in \{0, \ldots, k+1\}$ be such that $d'_i = c_{i-1}^{-1} d_i c_i$ with $c_0 = c_{k+1} = e$. Then

$$\operatorname{sign}(d_i) = \operatorname{sign}(c_{i-1}^{-1} d_i) = \operatorname{sign}(c_{i-1}^{-1} d_i c_i) = \operatorname{sign}(d'_i)$$

by Proposition 5.3.4 and "sign" either "sign_A" or "sign_B".

We claim that:

Lemma 5.4.1. *Let $G = A \star_C B$ and $\Phi\colon G \to \mathcal{A}$ be as above. Then Φ is a letter-quasimorphism.*

We will prove this by giving another description of Φ in terms of paths in the Bass–Serre tree associated to the amalgamated free product $G = A \star_C B$:

Let T be the tree with vertex set $\mathrm{V(T)} = \{gA \mid g \in G\} \sqcup \{gB \mid g \in G\}$ and oriented edges

$$\mathrm{E(T)} = \{(gA, gB) \mid g \in G\} \sqcup \{(gB, gA) \mid g \in G\} \subset \mathrm{V(T)} \times \mathrm{V(T)}$$

We define $\iota, \tau\colon \mathrm{E(T)} \to \mathrm{V(T)}$ via $\iota((gA, gB)) = gA$, $\tau((gA, gB)) = gB$ and similarly, $\iota(gB, gA) = gB$, $\tau(gB, gA) = gA$. Moreover, we set $(gA, gB)^{-1} = (gB, gA)$ and $(gB, gA)^{-1} = (gA, gB)$. It is well-known that T is indeed a connected tree.

G acts on T by left multiplication. We have that $\mathrm{Stab}_G(gA) = gAg^{-1} < G$, respectively $\mathrm{Stab}_G(hB) = hBh^{-1} < G$, $\mathrm{Stab}_G(gA, gB) = gCg^{-1}$ and $\mathrm{Stab}_G(gB, gA) = gCg^{-1}$

A *reduced path of edges* is a sequence $\wp = (e_1, \dots e_n)$, $e_i \in \mathrm{E(T)}$ such that $\tau(e_i) = \iota(e_{i+1})$ for every $i \in \{1, \dots, n-1\}$, without backtracking. We call n the *length of the path.* For what follows, $\mathcal{P}$ will be the set of all paths of edges.

We define the following map $\Xi\colon \mathcal{P} \to \mathcal{A}$ assigning an alternating word to each path of edges. Let $\wp \in \mathcal{P}$. If $\wp$ has length 1, then set $\Xi(\wp) := e$. Else, suppose that $\wp$ has length 2, i.e. $\wp = (e_1, e_2)$. Suppose that $e_1 = (g_1A, g_1B)$ and $e_2 = (g_2B, g_2A)$ and note that $g_1B = g_2B$. In particular, $g_1^{-1}g_2 \in B$. Set $\Xi(\wp) = \Xi((e_1, e_2)) = \mathtt{b}^{\mathrm{sign}_B(g_1^{-1}g_2)}$. Similarly, if $e_1 = (g_1B, g_1A)$ and $e_2 = (g_2A, g_2B)$ note that $g_1A = g_2A$ and set $\Xi(\wp) = \Xi((e_1, e_2)) = \mathtt{a}^{\mathrm{sign}_A(g_1^{-1}g_2)}$. Finally, for an arbitrary paths $\wp = (e_1, \dots, e_n)$ set $\Xi(\wp) = \Xi(e_1, e_2) \cdot \Xi(e_2, e_3) \cdots \Xi(e_{n-2}, e_{n-1}) \cdot \Xi(e_{n-1}, e_n)$. Note that Ξ is well defined. To see this, note that the stabilizer of any edge (gA, gB) (resp. (gB, gA)) is gCg^{-1}. Hence, if $(gA, gB) = (g'A, g'B)$ (resp. $(gB, gA) = (g'B, g'A)$) there is a $c \in C$ such that $gc = g'$. If (e_1, e_2) is a path of edges such that without loss of generality $e_1 = (g_1A, g_1B) = (g'_1A, g'_1B)$ and $e_2 = (g_2A, g_2B) = (g'_2A, g'_2B)$ then there are c_1, c_2 such that $g_1 = g'_1c_1$ and $g_2 = g'_2c_2$. Hence

$$\mathrm{sign}_B(g_1^{-1}g_2) = \mathrm{sign}_B(c_1^{-1}{g'_1}^{-1}g'_2c_2) = \mathrm{sign}_B({g'_1}^{-1}g'_2)$$

by Proposition 5.3.4. Define the *inverse of a path* $\wp = (e_1, \dots, e_n)$ as $\wp^{-1} := (e_n^{-1}, \dots, e_1^{-1})$. We see that $\Xi(\wp^{-1}) = \Xi(\wp)^{-1}$ using that $\mathrm{sign}(g^{-1}) = -\mathrm{sign}(g)$. We collect some further properties of Ξ. We note that if $\wp \in \mathcal{P}$ is a path then so is ${}^g\wp$, where ${}^g\wp$ denotes the image of $\wp$ under the action of $g \in G$.

Proposition 5.4.2. $\Xi\colon \mathcal{P} \to \mathcal{A}$ *has the following properties:*

(i) *For any* $\wp \in \mathcal{P}$ *and* $g \in G$ *we have* $\Xi({}^g\wp) = \Xi(\wp)$.

(ii) *Let* $\wp_1, \wp_2$ *be two paths of edges such that the last edge in* $\wp_1$ *is* e_1, *the first edge of* $\wp_2$ *is* e_2 *such that* $\tau(e_1) = \iota(e_2)$ *and such that* $e_1 \neq e_2^{-1}$. *Then* $\Xi(\wp_1 \cdot \wp_2) = \Xi(\wp_1)\Xi(e_1, e_2)\Xi(\wp_2)$ *as reduced words, where* $\wp_1 \cdot \wp_2$ *denotes the concatenation of paths.*

(iii) *Let* $g \in G$ *and let* $\wp(g)$ *be the unique path of edges from one of edges* $\{(A, B), (B, A)\}$ *to one of the edges* $\{(gA, gB), (gB, gA)\}$. *Then* $\Xi(\wp(g)) = \Phi(g)$, *for* Φ *as above.*

Proof. To see (i) note that for any path (e_1, e_2) with $e_1 = (g_1A, g_1B)$ and $e_2 = (g_2B, g_2A)$ we have

$$\Xi(e_1, e_2) = \mathsf{b}^{\mathrm{sign}(g_1^{-1}g_2)} = \mathsf{b}^{\mathrm{sign}(g_1^{-1}g^{-1}gg_2)} = \Xi({}^g(e_1, e_2))$$

and the same argument holds for paths with $e_1 = (g_1B, g_1A)$ and $e_2 = (g_2A, g_2B)$. Point (ii) is immediate from the definition.

To see (iii), without loss of generality assume that the normal form of g is $g = a_1b_1 \cdots a_kb_k$. Then

$$\wp(g) = (B, A), (a_1A, a_1B), (a_1b_1B, a_1b_1A), \ldots, (gB, gA)$$

and comparing $\Xi(\wp(g))$ with $\Phi(g)$ yields (iii). □

We can now prove Lemma 5.4.1:

Proof. Let $g, h \in G$. First, suppose that the midpoints of

$$\{(A, B), (B, A)\},\ \{(gA, gB), (gB, gA)\} \text{ and } \{(ghA, ghB), (ghB, ghA)\} \qquad (5.3)$$

lie on a common geodesic segment in T. If the midpoint of $\{(gA, gB), (gB, gA)\}$ lies in the middle of this segment then there are paths $\wp_1$ and $\wp_2$ such that $\wp(g) = \wp_1 \cdot e$, ${}^g\wp(h) = e \cdot \wp_2$ and $\wp(gh) = \wp_1 \cdot e \cdot \wp_2$ for e either (gA, gB) or (gB, gA). We see that in this case $\Xi(\wp_1 \cdot e) \cdot \Xi(e \cdot \wp_2) = \Xi(\wp_1 \cdot e \cdot \wp_2)$ as reduced words in $\mathcal{A}$ and hence $\Phi(g)\Phi(h) = \Phi(gh)$. Analogously we see that $\Phi(g)\Phi(h) = \Phi(gh)$ when the midpoint of $\{(A, B), (B, A)\}$ or $\{(ghA, ghB), (ghB, ghA)\}$ lies in the middle of this segment. Hence in this case Φ, $g, h \in G$ are as in (1) of Definition 5.2.1.

Now suppose that the midpoints in (5.3) do not lie on a common geodesic segment. Then there are non-trivial paths $\wp_1, \wp_2, \wp_3 \in \mathcal{P}$ with initial edges e_1, e_2, e_3 satisfying $\iota(e_1) = \iota(e_2) = \iota(e_3)$ and $e_i \neq e_j$ for $i \neq j$ such that

$$\wp(g) = \wp_1^{-1} \cdot \wp_2 \ , \ {}^{g}\wp(h) = \wp_2^{-1} \cdot \wp_3 \ , \text{ and } {}^{gh}\wp(((gh)^{-1}) = \wp_3^{-1} \cdot \wp_1.$$

By Proposition 5.4.2 we infer that

$$\begin{aligned} \Phi(g) &= c_1^{-1}\Xi(e_1^{-1}, e_2)c_2 \\ \Phi(h) &= c_2^{-1}\Xi(e_2^{-1}, e_3)c_3 \\ \Phi(gh)^{-1} &= c_3^{-1}\Xi(e_3^{-1}, e_1)c_1 \end{aligned}$$

for $c_i = \Xi(p_i)$, $i \in \{1, 2, 3\}$. Without loss of generality assume that $e_i = (g_iA, g_iB)$, the case $e_i = (g_iB, g_iA)$ is analogous. Then

$$\begin{aligned} \Phi(g) &= c_1^{-1}\mathtt{x}_1 c_2 \\ \Phi(h) &= c_2^{-1}\mathtt{x}_2 c_3 \\ \Phi(gh)^{-1} &= c_3^{-1}\mathtt{x}_3 c_1 \end{aligned}$$

$$\mathtt{x}_1 = \mathtt{b}^{\mathrm{sign}_B(g_1^{-1}g_2)}, \ \mathtt{x}_2 = \mathtt{b}^{\mathrm{sign}_B(g_2^{-1}g_3)}, \text{ and } \mathtt{x}_3 = \mathtt{b}^{\mathrm{sign}_B(g_3^{-1}g_1)}$$

We claim that $\mathrm{sign}_B(g_1^{-1}g_2) + \mathrm{sign}_B(g_2^{-1}g_3) + \mathrm{sign}_B(g_3^{-1}g_1) \in \{-1, +1\}$. To see this, note that all of the signs are either $\{+1, -1\}$ as the edges e_i were assumed to be distinct. Suppose that $\mathrm{sign}_B(g_1^{-1}g_2) = \mathrm{sign}_B(g_2^{-1}g_3) = \mathrm{sign}_B(g_3^{-1}g_1) = 1$.

Then $g_1^{-1}g_2C \succ C$, hence $g_3^{-1}g_2C = (g_3^{-1}g_1)g_1^{-1}g_2C \succ g_3^{-1}g_1C \succ C$, so $\mathrm{sign}_B(g_3^{-1}g_2) = 1$ and hence $\mathrm{sign}_B(g_2^{-1}g_3) = -1$, contradiction. Similarly, not all signs can be negative. Hence indeed $\mathrm{sign}_B(g_1^{-1}g_2) + \mathrm{sign}_B(g_2^{-1}g_3) + \mathrm{sign}_B(g_3^{-1}g_1) \in \{-1, +1\}$ and so $\mathtt{x}_1\mathtt{x}_2\mathtt{x}_3 \in \{\mathtt{b}, \mathtt{b}^{-1}\}$. This shows that Φ is as in (2) of Definition 5.2.1, hence Φ is a letter-quasimorphism. $\square$

Theorem F. *Let A, B, C be groups and $\kappa_A \colon C \hookrightarrow A$, $\kappa_B \colon C \hookrightarrow B$ be injections such that both $\kappa_A(C)$ and $\kappa_B(C)$ are left relatively convex subgroup of A resp. B. Let $G = A \star_C B$ be the amalgamated free product for this data. Then for every element $g_0 \in G$ which does not conjugate into A or B, there is a homogeneous quasimorphism $\bar{\phi} \colon G \to \mathbb{R}$ such that $\bar{\phi}(g_0) \geq 1$, $D(\bar{\phi}) \leq 1$ and $\bar{\phi}$ vanishes on A and B. If $g_0 \in G'$, then $\mathrm{scl}(g_0) \geq 1/2$.*

If G is countable then there is an action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1 \bar{\phi}] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}^2_b(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.

Remark 5.4.3. The methods developed in this paper may be modified to obtain similar gap results for HNN-extensions and graphs of groups, as well gap results for certain one-relator groups. A generalisation of this and direct proofs of these results using both quasimorphisms and surface mappings will appear in the forthcoming preprint [CH].

The existence of a uniform gap was known before; see [CF10] and Subsection 2.3.2

Proof. Let $g_0 \in G$ be as in the Theorem. Then, if g_0 does not conjugate into A or B we may conjugate g_0 by an element $g_1 \in G$ such that

$$g' = g_1 g_0 g_1^{-1} = a_1 b_1 \cdots a_k b_k$$

for *all of* $a_i \in A \smallsetminus \kappa_A(C)$ and $b_i \in B \smallsetminus \kappa_B(C)$. It follows that $\Phi(g') = w$ is a non-empty alternating word of even length and that $\Phi(g'^n) = w^n$ for $n \in \mathbb{N}$. By Theorem E there is a homogeneous quasimorphism $\bar{\phi} \colon G \to \mathbb{R}$ with $D(\bar{\phi}) \leq 1$ and $1 \leq \bar{\phi}(g_0) = \bar{\phi}(g')$ using that homogeneous quasimorphisms are invariant under conjugation. If G is countable then this quasimorphism $\bar{\phi}$ is moreover induced by a circle action $\rho \colon G \to \mathrm{Homeo}^+(S^1)$. $\square$

5.5 Right-Angled Artin Groups

In this section all graphs will be simplicial, i.e. do not contain multiple edges between two vertices or loops. Let Γ be a finite simplicial graph with vertices $\mathrm{V}(\Gamma)$ and edges $\mathrm{E}(\Gamma)$. Given a subset $\Lambda \subset \mathrm{V}(\Gamma)$ the *full subgraph on* Λ *in* Γ is the graph with vertices Λ where two elements $v, w \in \Lambda$ are connected by an edge if and only if they are connected in Γ.

For a vertex $v \in \Gamma$, the *link of* v is the full subgraph of the set $\{w \mid (v, w) \in \mathrm{E}(\Gamma)\}$ in Γ and denoted by $\mathrm{Lk}(v)$. The *closed star* is the full subgraph of $\mathrm{Lk}(v) \cup \{v\}$ in Γ and denoted by $\mathrm{St}(v)$. The *right-angled Artin group* or *RAAG* on Γ is the group $\mathrm{A}(\Gamma)$ with group presentation

$$\mathrm{A}(\Gamma) = \langle \mathrm{V}(\Gamma) \mid [v, w]; (v, w) \in \mathrm{E}(\Gamma) \rangle$$

A word w in the generators $\mathrm{V}(\Gamma)$ representing an element $[w] \in \mathrm{A}(\Gamma)$ is called *reduced* if it has minimal word length among all words representing $[w]$. A word w is said to be cyclically reduced if it has minimal word length among all of its conjugates. The *support* of an element $g \in \mathrm{A}(\Gamma)$ is the set of vertices that appear in a reduced word representing g. It is well-known that the support is well-defined.

Let Γ be a finite simplicial graph, let $\mathrm{A}(\Gamma)$ be the right-angled Artin group of Γ and let $v \in \Gamma$. Then $\mathrm{A}(\Gamma)$ can be thought of as an amalgamated free product of $\mathrm{A}(\mathrm{St}(v))$ and $A(\Gamma \backslash \{v\})$ where the common subgroup is $\mathrm{A}(\mathrm{Lk}(v))$. i.e.

$$\mathrm{A}(\Gamma) = \mathrm{A}(\mathrm{St}(v)) \star_{\mathrm{A}(\mathrm{Lk}(v))} \mathrm{A}(\Gamma \backslash \{v\}).$$

This will be used both in the proof of Theorem G and for induction arguments.

Proposition 5.5.1. *(Section 4 of [ADS15]) Let* $\Lambda \subset \Gamma$ *be a full subgraph of* Γ*. Then* $\mathrm{A}(\Lambda) < \mathrm{A}(\Gamma)$ *induced by the embedding, is a left relatively convex subgroup.*

Proof. We follow the proof of [ADS15]. We may induct on the following statement: For any Γ of size at most k and every full subgraph $\Lambda \subset \Gamma$, $\mathrm{A}(\Lambda)$ is left relatively convex in $\mathrm{A}(\Gamma)$. For $k = 2$ this is just the case of free-abelian and non-abelian free groups mentioned before. Assume the statement is true for all $n \leq k$. Let Γ be a graph with $k + 1$ vertices and let $\Lambda \subset \Gamma$ be a full subgraph. If $\Lambda = \Gamma$ there is nothing to show. Else pick $v \in \mathrm{V}(\Gamma) \backslash \mathrm{V}(\Lambda)$ and set Γ' to be the full subgraph in Γ on the vertices $\mathrm{V}(\Gamma) \backslash \{v\}$. Hence $\Lambda \subset \Gamma' \subset \Gamma$ with Γ' of size k. We wish to show that $\mathrm{A}(\Gamma') < \mathrm{A}(\Gamma)$ is a left-relatively convex subgroup. Consider the amalgamation

$$\mathrm{A}(\Gamma) = \mathrm{A}(\mathrm{St}(v)) \star_{\mathrm{A}(\mathrm{Lk}(v))} \mathrm{A}(\Gamma')$$

By induction, $\mathrm{A}(\mathrm{Lk}(v)) < \mathrm{A}(\Gamma')$ is a left relatively convex subgroup. Also $\mathrm{A}(\mathrm{Lk}(v)) < \mathrm{A}(\mathrm{St}(v))$ is a left relatively convex subgroup as $\mathrm{A}(\mathrm{St}(v)) = \langle v \rangle \times \mathrm{A}(\mathrm{Lk}(v))$.

We may use Corollary 5.3.3 to see that $\mathrm{A}(\Gamma') < \mathrm{A}(\Gamma)$ is a left relatively convex subgroup. By induction hypothesis, $\mathrm{A}(\Lambda) < \mathrm{A}(\Gamma')$ is a left-relatively convex subgroup and by transitivity $\mathrm{A}(\Lambda) < \mathrm{A}(\Gamma')$ is a left relatively convex subgroup. □

We deduce:

Theorem 5.5.2. *Let $g \in \mathrm{A}(\Gamma)$ be an element in an right-angled Artin group $\mathrm{A}(\Gamma)$ such that g_0 does not conjugate into a subgroup of a clique of Γ. Then there is a homogeneous quasimorphism $\bar{\phi}$ which vanishes on the generators $\mathrm{V}(\Gamma)$ such that $\bar{\phi}(g_0) \geq 1$ and $D(\bar{\phi}) \leq 1$.*

Moreover, there is an action $\rho \colon \mathrm{A}(\Gamma) \to \mathrm{Homeo}^+(S^1)$ such that $[\delta^1 \bar{\phi}] = \rho^* \mathrm{eu}_b^{\mathbb{R}} \in \mathrm{H}_b^2(G, \mathbb{R})$, for $\mathrm{eu}_b^{\mathbb{R}}$ the real bounded Euler class.

Observe that no non-trivial element in the commutator subgroup of a right-angled Artin group conjugates into a clique. An application of Bavard's Duality Theorem 2.3.2 yields:

Theorem G. *Let g_0 be a non-trivial element in the commutator subgroup of a right-angled Artin group. Then $\mathrm{scl}(g_0) \geq 1/2$. This bound is sharp.*

Proof. (of Theorem 5.5.2) Let $g \in \mathrm{A}(\Gamma)$ be such an element. We may suppose that g is cyclically reduced, as homogeneous quasimorphisms are invariant under conjugation. Choose a vertex v in the support of g such that there is another vertex w in the support of g which is non-adjacent to v. Such a vertex exists as g does not conjugate into a clique. Write $\mathrm{A}(\Gamma)$ as

$$\mathrm{A}(\Gamma) = \mathrm{A}(\mathrm{St}(v)) \star_{\mathrm{A}(\mathrm{Lk}(v))} \mathrm{A}(\Gamma \backslash \{v\})$$

and observe that g does not conjugate into any factor of this amalgamation as both v and w are in the support of g. By Proposition 5.5.1, both $\mathrm{A}(\mathrm{Lk}(v)) < \mathrm{A}(\mathrm{St}(v))$ and $\mathrm{A}(\mathrm{Lk}(v)) < \mathrm{A}(\Gamma \backslash \{v\})$ are left relatively convex subgroups. We conclude using Theorem F. Commutators in $\mathrm{A}(\Gamma)$ have scl at most $1/2$. Hence this bound is sharp. □

Chapter 6

Collaborative work and on-going projects

Here we describe open problems and topics that are the subject of collaborative, on-going, or future research projects.

6.1 Spectrum of Simplicial Volume

The simplicial volume $\|M\|$ of an orientable closed connected manifold M is a homotopy invariant that captures the complexity of representing the fundamental class by singular cycles with real coefficients (see Section 2.4 for a precise definition and basic terminology). Simplicial volume is known to be positive in the presence of enough negative curvature [Gro82] and known to vanish in the presence of enough amenability. It provides a topological lower bound for the minimal Riemannian volume (suitably normalised) in the case of smooth manifolds [Gro82].

Until now, for large dimensions d, very little was known about the precise structure of the set $\mathrm{SV}(d) \subset \mathbb{R}_{\geq 0}$ of simplicial volumes of orientable closed connected d-manifolds. The set $\mathrm{SV}(d)$ is countable and closed under addition; see Section 2.4. However, the set of simplicial volumes is fully understood only in dimensions 2 and 3 with $\mathrm{SV}(2) = \mathbb{N}[4]$ (Example 2.4.2) and $\mathrm{SV}(3) = \mathbb{N}[\frac{\mathrm{vol}(M)}{v} \mid M]$, where M ranges over all complete finite-volume hyperbolic 3-manifolds with toroidal boundary and where $v > 0$ is a constant (Example 2.4.3).

This reveals that there is a *gap* of simplicial volume in dimensions 2 and 3: For $d \in \{2, 3\}$ there is a constant $C_d > 0$ such that the simplicial volume of an orientable closed connected d-manifold either vanishes or is at least C_d. It was an open question [Sam99, p. 550] whether such a gap exists in higher dimensions. For example, until now the lowest known simplicial volume of an orientable closed connected 4-manifold has been 24 [BK08] (Example 2.4.4).

In joint work with Clara Löh we could show that dimensions 2 and 3 are the *only* dimensions with such a gap.

Theorem I (Theorem A; [HL19]). *Let $d \geq 4$ be an integer. For every $\epsilon > 0$ there is an orientable closed connected d-manifold M such that $0 < \|M\| \leq \epsilon$. Hence, the set of simplicial volumes of orientable closed connected d-manifolds is dense in $\mathbb{R}_{\geq 0}$.*

In dimension 4, we could get the following refinement of Theorem I.

Theorem J (Theorem B; [HL19]). *For every $q \in \mathbb{Q}_{\geq 0}$ there is an orientable closed connected 4-manifold M_q with $\|M_q\| = q$.*

We prove these statements by relating simplicial volume to stable commutator length. We show:

Theorem K (Theorem F; [HL19]). *Let G be a finitely presented group that satisfies $\mathrm{H}_2(G, \mathbb{R}) \cong 0$ and let $g \in G'$ be an element in the commutator subgroup. Then there is an orientable closed connected 4-manifold M_g with*

$$\|M_g\| = 48 \cdot scl(g).$$

6.2 Computational Complexity of Commutator Length

Let F be a non-abelian free group on finitely many generators. Consider the following decision problems for (stable) commutator length.

Problem (SCL-F).

INPUT: $w \in F'$, $k \in \mathbb{N}$.

OUTPUT: *Is* $\mathrm{scl}(w) \leq k$*?*

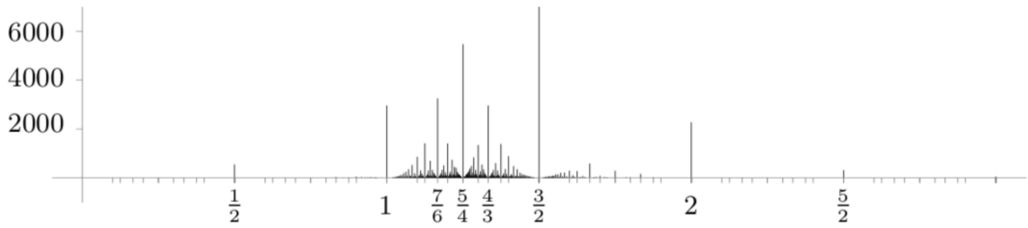

Figure 6.1: scl histogram for 50000 alternating words of length 36 as in [Cal09b].

where $\mathrm{scl}(w)$ denotes the stable commutator length of w in F and

Problem (CL-F).

INPUT: $w \in F'$, $k \in \mathbb{N}$.

OUTPUT: *Is* $\mathrm{cl}(w) \leq k$*?*

where $\mathrm{cl}(w)$ denotes the commutator length of w in F. Both of these decision problems are decidable as shown by Calegari in [Cal09b] and Culler in [Cul81]. Indeed, Calegari showed that *stable* commutator length maybe be computed in polynomial time i.e. **SCL-F** is in the complexity class P.

Conjecture 6.2.1. **CL-F** is NP-complete.

6.3 Open Questions in Stable Commutator Length

There are many open questions concerning stable commutator length in free groups.

1. Explain the histogram of Figure 6.1: How are the values of scl distributed and which rational numbers are obtained?

2. In particular, is there a "second gap" for free groups, i.e. is there a constant $C > 1/2$ such that every non-trivial $f \in F'$ satisfies $\mathrm{scl}(f) = 1/2$ or $\mathrm{scl}(f) \geq C$. For this, one may wish to characterise the elements for which $\mathrm{scl}(f) = 1/2$.

   Conjecture 6.3.1. Let $f \in F'$ be such that $\mathrm{scl}(f) = 1/2$. Then either $\mathrm{cl}(f) = 1$ or there is a $t \in F$ such that $\mathrm{cl}(ftft^{-1}) = 1$.

In joint work with Giles Gardam we could verify this conjecture for all elements in the free group on two generators of length less than 20.

3. Recall that in a non-abelian free group F an element $w \in F$ is called *primitive* if it is part of a free basis of F. For an element $w \in F$ define the *primitivity rank* of $\pi(w)$ as

$$\pi(w) = \min\{rk(K) \mid w \in K < F \text{ and } w \text{ not primitive in } K\},$$

where by convention $\pi(w) = \infty$ if and only if $w \in F$ is primitive, as in this case w is also primitive in every subgroup of F containing it. The primitivity rank was introduced by Puder [Pud14] and has been used by Louder and Wilton [LW18] to study negative immersions in one-relator groups. In light of Conjecture 6.4.1 we believe that there is a connection between surface maps to one-relator groups and the stable commutator length of the relator in the free group. Computer experiments suggest the following relationship between the primitivity rank and stable commutator length:

Conjecture 6.3.2. Let $w \in F$ be an element in the free group. Then

$$\mathrm{scl}(w) \geq \frac{\pi(w) - 1}{2}.$$

This would generalise the gap found by Duncan and Howie [DH91] and would show that a 'second gap' only occurs in the free group on two generators.

4. Is there a way to construct extremal quasimorphisms for arbitrary elements in F, analogous to the construction of Chapter 5? Are all such extremal quasimorphisms induced by circle actions?

6.4 Simplicial Volume of one-relator groups

Let F be freely generated by the finite set $\mathcal{S}$ and let $r \in F'$. Let $G_r = \langle \mathcal{S} \mid r \rangle$ be the one-relator group with defining relation r. Let α_r be the generator of $\mathrm{H}_2(G_r, \mathbb{Z})$ and let $\|\alpha_r\|_1$ be the ℓ^1-norm of α.

Conjecture 6.4.1. (Heuer–Löh) Let r and α_r be above. Then

$$\frac{1}{4}\|\alpha_r\|_1 - \frac{1}{2} = \mathrm{scl}(r).$$

Proposition 6.4.2. *(Heuer–Löh) The conjecture holds if*

- $r = r_1 \boldsymbol{x} r_2 \boldsymbol{x}^{-1} \in F(\mathcal{S} \cup \boldsymbol{x})$, *where* $r_1, r_2 \in F(\mathcal{S})$, *or*
- $r = r_1 r_2$, *where* $r_1 \in F(\mathcal{S}_1)$, $r_2 \in F(\mathcal{S}_2)$ *and* $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

Moreover, this conjecture has been computationally verified in many other cases.

This is joint work with Clara Löh.

6.5 Norms in Bounded Cohomology

There are two different norms on higher dimensional bounded cohomology of the free group $\mathrm{H}_b^n(F, \mathbb{R})$, $n \geq 3$ as defined in [FFPS19]. Recall that for a function $\alpha \in C_b^n(F, \mathbb{R})$, $\|\alpha\|_\infty$ denotes the supremums norm. For any subset $S \subset F$ let $\|\alpha\|_S = \{\|\delta\phi\|\infty \mid \phi(S) = 0, [\delta\phi] = \alpha\}$ and define $\|\alpha\|_{\infty,0} = \sup\{\|\alpha\|_S, S \subset F, S \text{ finite}\}$.

Conjecture 6.5.1. If $n \geq 3$ then the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{\infty,0}$ on $\mathrm{H}_b^n(F, \mathbb{R})$ are equivalent, i.e. there is a constant $C_n > 0$ such that for every class $\alpha \in \mathrm{H}_b^n(F, \mathbb{R})$ we have

$$\frac{1}{C_n}\|\alpha\|_\infty \leq \|\alpha\|_{\infty,0} \leq C_n\|\alpha\|_\infty.$$

6.6 Quasi-BNS Invariants

Let G be a group which is finitely generated by a set S. Let $Cay(G, S)$ be the Cayley graph of G with respect to S. The *first BNS-invariant* [BNS87] $S^1(G) \subset \mathrm{Hom}(G, \mathbb{R})$ is defined by setting

$$S^1(G) = \{\phi \in \mathrm{Hom}(G, \mathbb{R}) \mid G_\phi \subset Cay(G, S) \text{ is connected}\}$$

where $G_\phi = \{g \in G \mid \phi(g) > 0\}$. The study of these invariants is intimately linked to finiteness properties of subgroups of G that contain G'. However, we note that for random groups G, the homomorphism group $\mathrm{Hom}(G, \mathbb{R})$ is trivial and hence $S^1(G)$ is not a useful invariant. On the other hand random groups have many *quasimorphisms*, indeed this space is almost surely uncountable dimensional.

Question 6.6.1. Is there an analogous definition for BNS invariants involving quasimorphisms? How much of the original theory can be generalised to this case?

This is joint work with Dawid Kielak.

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